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Probability of forming a through conductive inclusion for the case of large-scale cylindrical inserts with their uniform distribution over the layer height and over the sample height

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For a special case of percolation, when the size of the inserts is comparable to the size of the system, the dependencies $P(r)$ of probability of forming a through inclusion are obtained from the radius of inserts — cylinders of radius r and height h. This form allows one to find $P(r)$ using the results obtained earlier for the layered case — the distribution of inserts over layers of height h in the sample from N layers with the number of inserts per layer n. The cases of homogeneous distribution of Nn inserts over height of sample of height Nh and a transitional case between given and layered are studied. For $n = 1$ the results were obtained analytically. For $N = 2$ and/or $n = 2$ results obtained analytically for $r = R$. For the general case, the results are obtained numerically. It is shown, that $P(r)$ can be described by a small number of numerical parameters: the value $P(R)$, percolation radius r_c and an indicator of proximity to the threshold form t_c. The dependences of the values of these parameters on N, n are obtained and analized.

Keywords: conducting inclusion, random insertions, transition from layered to three-dimensional, quasi-three-dimensional, quasi-one-dimensional, percolation.

Імовірність утворення наскрізного провідного включення для великомасштабних циліндричних вставок з рівномірним їх розподілом по висоті шару та по висоті зразка. *Р.Є.Бродський*

Для окремого випадку перколяції, коли розмір вставок порівнянний з розміром системи, отримані залежності $P(r)$ ймовірності утворення наскрізного включення від радіуса вставок — циліндрів радіуса r і висоти h. Ця форма дозволяє знайти $P(r)$. Використовуючи резулбтати отримані раніше для шаруватого випадку — розподілу вставок по шарах з висотою h у зразку з N шарів з кількістю вставок на шар n, вивчаються випадки однорідного розподілу Nn вставок по висоті зразка висоти Nh та перехідний випадок між даним та шаруватим. Для $n = 1$ результати були отримані аналітично. Для $N = 2$ та/або $n = 2$ результати отримані аналітично для $r = R$. Для загального випадку результати отримані чисельно. Показано, що $P(r)$ можна описати невеликою кількістю числових параметрів: величиною $P(R)$, радіусом перколяції r_c та показником близькості до порогової форми t_c. Отримано та проаналізовано залежності значень цих параметрів від N,n.

1. Introduction

If randomly distributed conductive inserts are present in a non-conductive medium, such inserts can form a through channel from one system boundary to another, thereby changing the properties of the system as a whole — the non-conductive system will become conductive. Conductivity can be understood not only as electrical conductivity, but also as "conductivity" of liquids or gases, i.e. formation of channels for the flow of a liquid or gas, for example, in a porous, powdery medium or ceramics.

The formation of through conductive inclusions in media with random conductive inserts is studied in the theory of percolation. The two main directions of the theory are discrete and continuous percolation. Both directions of the theory are actively developed. Both directions can refer not only to physical systems, but also to biological, social or informational ones. Works [1–3] are devoted to the spread, discrete percolation, of infection during epidemics and its blocking by immunization. Works [4–7] are devoted to the percolation of information in social networks and the Internet. Work [8] considers three-dimensional continuous percolation in a system where one of the axes is time, and the other two are the area where mobile wireless transmitters can appear and leave. Works devoted to physical percolation are actively published: electrical conductivity in composites [9–11], gas percolation in gas sensors [12–13], water in wood [14].

As a rule, in the theory of percolation, the case of a sufficiently large number of sufficiently small insertions is considered, while the system can be considered as conditionally unlimited and edge effects can be neglected. This allows one to obtain a number of general patterns. However, it is also of interest to study the opposite case, when the inserts have a size comparable to the size of the system, since this case can also be realized in real systems. In this case, the size of the sample (usually the ratio of the dimensions of the inserts and the sample) and the shape are of decisive importance for the probability of the appearance of a through conductive inclusion.

In [15], the formation of a through conductive inclusion for a layered system with continuous layers with round conductive inserts (islands) was studied — the problem is continuous in the plane of the layers and discrete in the direction perpendicular to the layers. The layered system was studied for the case of a small number of inserts, starting from one insert per layer, with a size comparable to the layer size. The same case is considered in the present work.

In this work, we study a system with the same cylindrical inserts, but located not in layers, i.e. at discrete levels of height, but distributed over the entire height of the sample. The case of a uniform distribution of conductive inserts over the sample height and the transitional case between homogeneous and layered are considered, when each layer contains exactly the same number of inserts as in the layered case, but they are distributed over the layer height.

Percolation in a system with cylindrical inserts was studied in [8, 16–20], but for the case of a large number of small inserts.

Consideration of large-scale insertions will make it possible to obtain analytical results for individual cases and compare with them the results obtained numerically for the general case.

1. Formulation of the problem

A cylindrical non-conductive sample with cylindrical conductive inserts is considered. All inserts are the same, the axes of the inserts are parallel to the axis of the sample. Inserts radius — r, sample radius — R, insert height — h. The sample height is assumed to be Nh, i.e. N — number of layers of height h in the sample. The sample is considered as vertical, i.e. we will talk about the lower and upper faces of the sample, the location of the inserts along the height of the sample. Inserts can intersect, and if the insert is close to the border of the sample (side or top/bottom face), cut off by the border. By contacting each other, the inserts can form a through, from the first to the last layer, a conductive inclusion. The probability of this event is the subject of study in this work.

The position of the insert is determined by its coordinates along the sample height and in the section plane. The insert position is the position of its center. The distribution of inserts in the section plane is random and uniform.

In [15], the case of a layered arrangement of inserts along the height of the sample was studied, i.e. on the heights $y_\nu = h/2 + \nu h$, $\nu = 0, \ldots, N-1$. At each height — the same number n of inserts.

Two cases are considered in this paper:

a. Inserts are located by height in intervals $[y_\nu - \Delta,\ y_\nu + \Delta)$, $\Delta \le h/2$, in each interval n inserts, distributed on height in intervals randomly and uniformly. I.e. inserts "shift" in the interval $\pm\Delta$ from their positions in the layered case.

b. $N \cdot n$ inserts are randomly and uniformly distributed over the height of the entire sample.

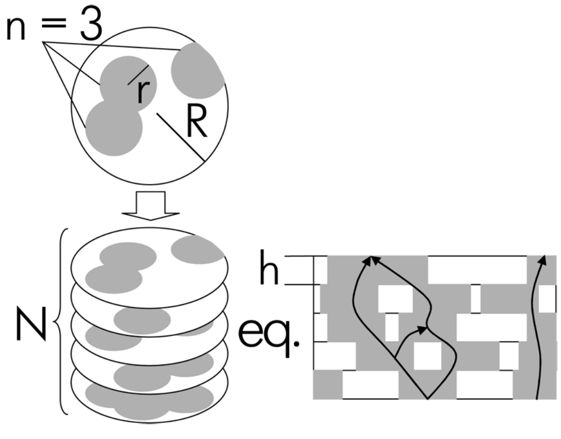

Fig.1. Schematic representation of the system under consideration for the layered case, the arrangement of islands in layers and the formation of a through conductive inclusion.

In the latter case, the inserts are uniformly distributed over the entire available three-dimensional area — the volume of the sample, such a case will be called three-dimensional. Case a. is similar to the three-dimensional case at $\Delta = h/2$. In both cases, the average number of insertions in any interval dy of sample height does not depend on y and equal $dy \cdot n/h$. However, the distribution of inserts along the height is different, in particular, the number of inserts in different layers of the sample, i.e. in intervals $[\nu h,(\nu+1)h)$ at this case strictly equal to n, and in the three-dimensional case it is random and different, and only their average number is equal to n. Such a case will be called quasi-three-dimensional. Case a. for arbitrary Δ will be called the case with displacements.

The distribution of inserts in the sample for the layered case is shown in Fig. 1, for the cases considered here in Fig. 2. It can be seen that at $\Delta < h/2$ the distribution of islands over the layers is still noticeable, in the quasi-three-dimensional case the distribution over the layers is no longer visible, but in each layer there are exactly $n = 3$ islands, finally, in the three-dimensional case, in different layers there are already a different number of islands — in some there are more than n at the expense of others.

In the layered case, it was considered that a through inclusion is formed if the sample contains a connected conducting inclusion containing at least one island in the first and at least one island in the last layer. We will retain the same definition here as well, assuming that an island is in the first layer if its center lies in the interval $[0, h)$ vertically, and is in the last one if it lies in $[(N-1)h, Nh)$.

2. Some general properties of the cases under consideration

Note that for large n in the three-dimensional case, the relative deviation $1/\sqrt{n}$ in the number of inserts in different layers from their average value tends to zero, so that the distribution of inserts in height will differ little from the quasi-three-dimensional one. Therefore, all properties of the system, including the probability $P(r)$ at $n \to \infty$ for these two cases will converge.

We suppose that in case a. even at maximum $\Delta = h/2$ only islands in neighboring layers (and the same layer) can have contact, there are no contacts through the layer.

In order for there to be contact between islands in adjacent layers in the layered case, it is sufficient that the distance between their centers in the section plane does not exceed $2r$. In the case of displacements, this condition must be satisfied, and it is also necessary that the islands "reach" each other vertically. Because the distributions of islands in the section plane and vertically are independent, the probability of contact in a given pair of islands is the product of the probabilities of such events. I.e. if at a certain arrangement of inserts in the case of displacements, a through inclusion is formed, then in the layered case, with the same arrangement of inserts in the plane of the layers, the inclusion will be obligatory, but not vice versa. The probability of forming a through inclusion in the case with displacements does not exceed the prob-

Fig.2. Schematic representation of the arrangement of islands for the case of displacements at Δ= 0.3 h/2 (left), for the quasi-three-dimensional case Δ=h/2 (center), for 3D case (right).

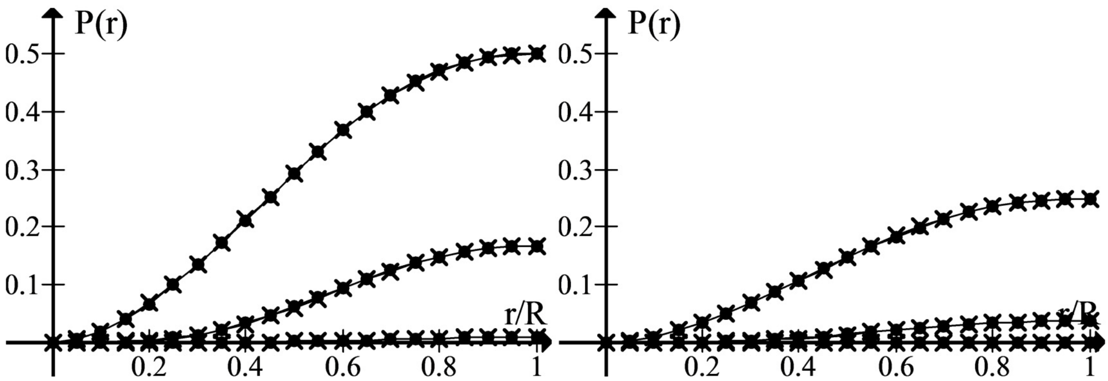

Fig.3. Probability P(r) of formation of a through conductive inclusion at n=1 island per layer for quasi-three-dimensional (left) and three-dimensional (right) cases. The crosses show the values $P^l(r)$ for the layered case, divided by N! left and N^N right.

ability for the layered case with the same parameters.

The probability that two islands in adjacent layers "reach" each other vertically is equal to the probability that the displacement of an island in the upper layer $\delta_\wedge \in [-\Delta, \Delta)$ no more than the displacement $\delta_\vee$ of island in the lower layer. The probability of this is equal to the probability of fulfilling the relation $\frac{\delta_\wedge}{\Delta} \le \frac{\delta_\vee}{\Delta}$, where the values on the right and left are uniformly distributed in $[-1, 1)$. This probability does not depends on Δ. That is, the probability of contact in a given pair of islands does not depend on Δ. This means that the probability of forming a through inclusion for the case of displacements is the same for any $\Delta > 0$. In what follows, for convenience, we will talk about the case $\Delta = h/2$, but all received $P(r)$ will be the same for any Δ for the case of displacements.

3. Some exactly solvable cases

The only island in the layer, quasi-three-dimensional case.

At $n = 1$ for the formation of a through inclusion, it is necessary that all islands enter into it. To do this, as mentioned above, it is necessary that in all adjacent layers the islands "reach" each other vertically and be at a distance not exceeding $2r$ in the section plane. The first condition is satisfied if in each pair of adjacent layers the displacement of the upper island does not exceed the displacement of the lower one. For the convenience of calculations, we pass from the displacements $\delta su\nu$ to locations $l_\vee = \delta_\vee + \Delta \in [0, 2\Delta)$. The probability of fulfilling the specified condition is

$$p_N = \int_0^h \frac{dl_1}{h} \int_0^{l_1} \frac{dl_2}{h} \int_0^{l_2} \frac{dl_3}{h} \quad \int_0^{l_{N-2}} \frac{dl_{N-1}}{h} \int_0^{l_{N-1}} \frac{dl_N}{h} = \frac{1}{N!}. \quad (1)$$

The probability of fulfilling the second condition coincides with the probability $P^l(r)$ of formation of a through inclusion in the layered case at the same N, r. So

$$P(r) = p_N P^l(r) = \frac{P^l(r)}{N!}. \quad (2)$$

For sample from $N = 2$ layers the probability that the centers of two islands in adjacent layers, i.e. two points uniformly distributed in a circle, are at a distance not exceeding $2r$ in the section plane is [21, 15]

$$P_2^l(r) =$$
$$= \frac{2}{\pi}\left(\frac{\pi}{2} - \left(1 - 4(\frac{r}{R})^2\right)\arccos\frac{r}{R} - \frac{r}{R}\left(1 + 2(\frac{r}{R})^2\right)\sqrt{1 - (\frac{r}{R})^2}\right).$$

If such a probability for each pair of layers from N did not depends on the contact in other layers, for N layers would be performed

$$P^l(r) = \left(P_2^l(r)\right)^{N-1}. \quad (3)$$

This equality holds exactly in the absence of edge effects, for layers — nested spheres or for the one-dimensional problem of layers-circles with inserts-arcs [22]. For round layers, expression (3) does not holds exactly, but as shown in [15], its values are very close to the results of a numerical experiment.

In Fig. 3 left values $P(r)$ are shown for the quasi-three-dimensional case at $n = 1$

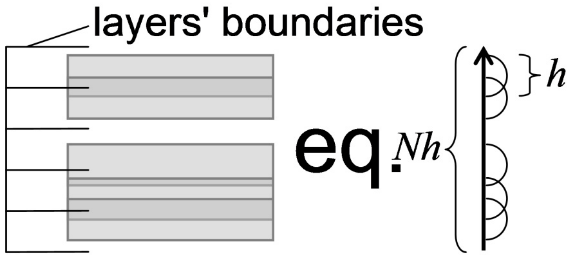

Fig.4. Schematic representation of the arrangement of islands in the one-dimensional limit r=2R on the left and the equivalent one-dimensional system on the right, the case of n=1 island per layer shown.

for $N = 2$, $N = 3$ and $N = 5$ obtained from the numerical experiment, as well as the values $P^l(r)$ from a numerical experiment with those parameters divided by $N!$. In the same place, solid lines show graphs (3) divided by $N!$.

3D case n = 1.

In the three-dimensional case $n = 1$ corresponds to N islands of height h in the sample of height Nh. Although in the three-dimensional case the distribution of islands over layers is not specified, to formulate the condition for the existence of a through inclusion, it is convenient to speak about N layers of height h. To form a through inclusion, i.e. an inclusion containing at least one island in the lower and upper layers, with N islands in the sample it is need in each layer of height h there was one island, and herewith there was a contact in each neighboring pair.

The probability of the second event is found above (2). The probability that N islands, uniformly distributed along the vertical of the sample, will be located one by one in N layers, is equal to

$$p_{N/N} = \left(\frac{1}{N}\right)^N \cdot N!. \quad (4)$$

So the probability of forming a through inclusion for the case under consideration is the product of (2) and (4)

$$P(r) = \frac{P^l(r)}{N^N}. \quad (5)$$

In Fig. 3 on the right the values $P(r)$ are shown for the three-dimensional case with $n = 1$ for $N = 2$, $N = 3$ and $N = 5$ obtained from the numerical experiment, as well as the values $P^l(r)$ from a numerical experiment with those parameters divided by

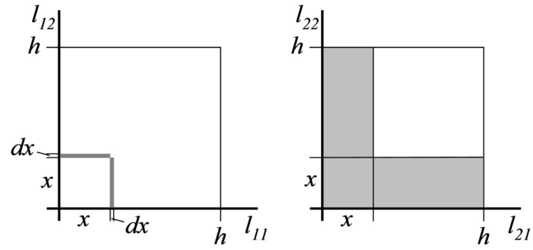

Fig. 5. Values of island locations at which a through inclusion is formed for the case $N = 2$, $n = 2$ suitable values are shown in gray for bottom layer left and top right.

N^N. In the same place, solid lines show plots (3) divided by N^N.

One-dimensional limit.

At $r = 2R$ the conductive phase of the insert occupies the entire cross-sectional area of the sample, even if the center of the island lies on the cross-section boundary, the problem becomes equivalent to a one-dimensional one, Fig. 4. The formation of a through inclusion corresponds to the filling of the segment-carrier with segments-inserts from the first to the last interval of height h without gaps.

However, the problem of forming a through conductive inclusion becomes equivalent to one-dimensional already at $r = R$, because at $r \geq R$ each island has contact with all those islands, as at $r = 2R$. The probability of forming a through inclusion for such a one-dimensional problem is the limit probability $P(r)$ at $r \to R$ for the same distribution of islands with the same N, n. In the layered case, this probability is equal to one. Let us find such a probability for some cases of a quasi-three-dimensional problem.

Let's start with the case N = 2; n = 2.

In order for a through inclusion to be formed, it is sufficient that one, any, island of the lower layer has contact with any island of the upper one. Let the top island of the bottom layer have the location l in the interval $(x, x + dx)$ and the other below. The probability of this is $2\frac{dx}{h}\frac{x}{h}$. The probability that at least one of the islands in the upper layer has a location $l \leq x$, is equal to one minus the probability that none of the islands has such an arrangement, i.e. is equal to $1-\left(\frac{h-x}{h}\right)^2$.

For clarity and ease of transition to large n suitable arrangements of islands are illus-

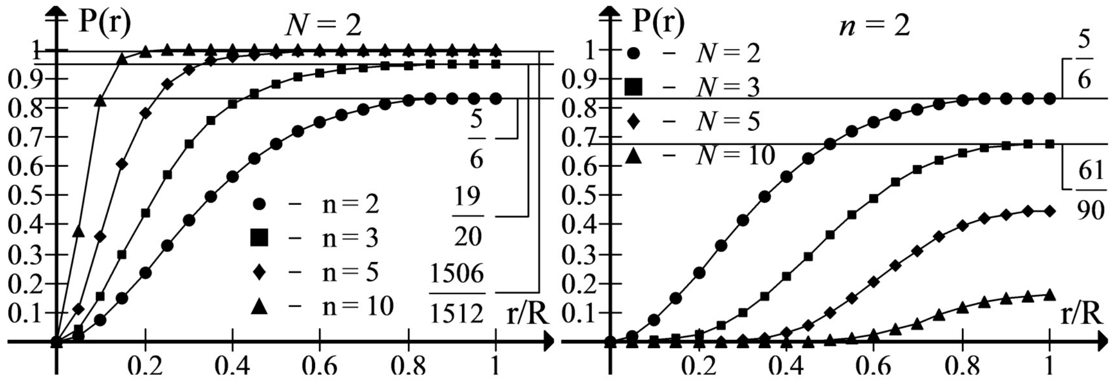

Fig. 6. Probabilities $P(r)$ for case N = 2 and various n left and case n = 2 and various N right. Analytically found values $P(R)$ are shown.

trated in Fig. 5, left — for the bottom layer, right — for the top. Locations l_{ij} of j-th island in i-th layer are plotted along the axes.

The probability of forming a through inclusion is equal to

$$P = \int_0^h 2\frac{dx}{h}\frac{x}{h}\left(1-\left(\frac{h-x}{h}\right)^2\right), \tag{6}$$

$$P = 2\left(\frac{2}{3}-\frac{1}{4}\right) = \frac{5}{6}.$$

Intermediate calculations are shown to compare with the following case.

For the case $N = 2$ and arbitrary n the reasoning is similar. (The illustration of Fig. 5 needs to be expanded to n dimensions.)

$$P = \int_0^h n\frac{dx}{h}\left(\frac{x}{h}\right)^{n-1}\left(1-\left(\frac{h-x}{h}\right)^n\right) = \tag{7}$$

$$= n\sum_{a=0}^{n-1}(-1)^{(n-a+1)}C_n^a\frac{1}{2n-a}.$$

At n = 2 the resulting expression reduces to $P = 2\left(\frac{-1}{4}+2\frac{1}{3}\right)$, i.e. to the expression above. At n = 1 expression is reduced to a single term $P = \frac{1}{2}$, i.e. $\frac{1}{N!}$ at N = 2 according to (2).

Table

n	P
2	5/6
3	19/20
5	1506/1512

For the first ones studied further in the numerical experiment n

As you can see, the probability quickly tends to unity with increasing of n, as expected.

The obtained values are limiting at $r \to R$. Graphs $P(r)$ for N = 2, n = 2, 3, 5, 10 for the quasi-three-dimensional case, obtained from a numerical experiment, are shown in Fig. 6 on the left. In the same place, the horizontal lines show the values P for $r \geq R$ found above. As can be seen, at $r \to R$ graphs tend to the corresponding values.

At **n = 2 and arbitrary N** to form a through inclusion islands in each ν-th layer, except for the first and last, must have contact with both the previous and the next layer. In order for there to be contact with the previous layer, it is necessary, as above, that at least one island has the location $l \leq x_{\nu-1}$. And to determine the presence of contact with the next, as above, we will integrate over the position x_ν of the top island of the current layer. The joint probability of both locations depends on the ratio of $x_{\nu-1}$ and x_ν (in the illustration of Fig. 5, a suitable location is the intersection of the areas of Fig. on the left for $x = x_\nu$ and right for $x = x_{\nu-1}$). If $x \leq x_{\nu-1}$, the probability is $2\frac{dx_\nu}{h}\frac{x_\nu}{h}$, if $x_\nu > x_{\nu-1}$, it is equal to $2\frac{dx_\nu}{h}\frac{x_{\nu-1}}{h}$. So for N layers

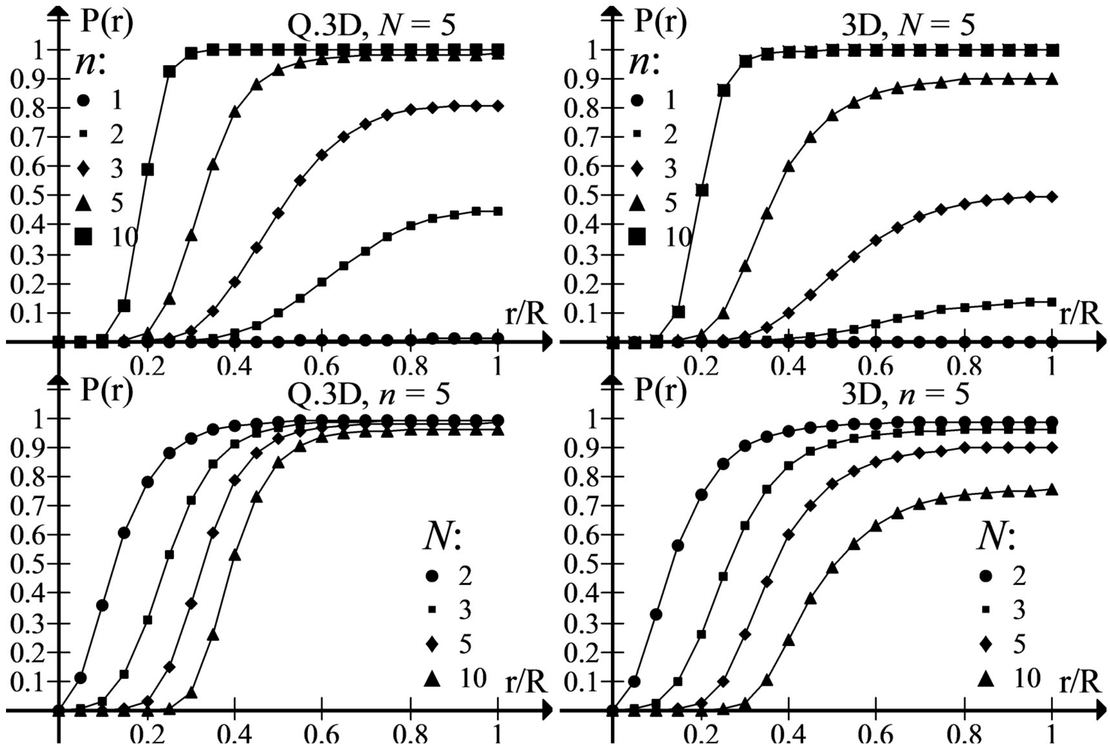

Fig. 7. Probabilities $P(r)$ for arbitrary N, n examples. For various n at one N top, for different N at one n bottom. Left for quasi-3D, right for 3D.

$$P = \int_0^h 2\frac{dx_1}{h}\frac{x_1}{h} \cdot \int_0^h 2\frac{dx_2}{h}\begin{cases}\frac{x_2}{h}, x_2 \le x_1\\ \frac{x_1}{h}, x_2 > x_1\end{cases} \cdot \ldots \quad (8)$$

$$\cdot \ldots \int_0^h 2\frac{dx_{N-1}}{h}\begin{cases}\frac{x_{N-1}}{h}, x_{N-1} \le x_{N-2}\\ \frac{x_{N-2}}{h}, x_{N-1} > x_{N-2}\end{cases} \cdot \int_0^h 2\frac{dx_N}{h}\begin{cases}\frac{x_N}{h}, x_N \le x_{N-1}\\ \frac{x_{N-1}}{h}, x_N > x_{N-1}\end{cases}.$$

The integral for the last layer is $2\frac{x_{N-1}}{h} - \left(\frac{x_{N-1}}{h}\right)^2$, which matches $1 - \left(\frac{h-x}{h}\right)^2$ (6).

For $N = 3$ integral (8) is equal to $\frac{61}{90}$ that is less than $\frac{5}{6}$ for $N = 2$. With an increase of N the probability, as expected, decreases and tends to zero.

Graphs $P(r)$ for $n = 2$, $N = 2$, 3, 5, 10 for the quasi-three-dimensional case, obtained from a numerical experiment, are shown in Fig. 6 right. In the same place, the horizontal lines show the values P at $r \geq R$ found above for $N = 2$, 3. As can be seen, at $r \to R$ graphs tend to the corresponding values.

General case of arbitrary N, n

Case $N > 2$, $n > 2$ investigated by numerical simulation. Dependencies $P(r)$ were obtained for quasi-3D and 3D distribution of islands for $N = 2$; 3; 5; 10 and $n = 1$; 2; 3; 5; 10 (for the three-dimensional case — $N \times n$ islands in sample of height Nh).

For all N, n general view of graphs $P(r)$ and the nature of the changes of $P(r)$ with N and n are same. On Fig. 7 for example, a series of graphs $P(r)$ are presented for various n at one $N = 5$ (top) and for various N at one $n = 5$ (bottom), for quasi-three-dimensional (left pair) and three-dimensional (right pair) cases.

As can be seen, for all N, n dependencies $P(r)$ have an S-shape. At the same N graphs are arranged in ascending order of n from bottom to top, or, equivalently, from right to left. With one n graphs $P(r)$ arranged in ascending order of N from left to right or, equivalently, from top to bottom. I.e. for the same island radius r probability $P(r)$ increases with the number of islands n and decreases with increasing of number of layers N, which is to be expected, there are no intersections of graphs, the order for all r is the same.

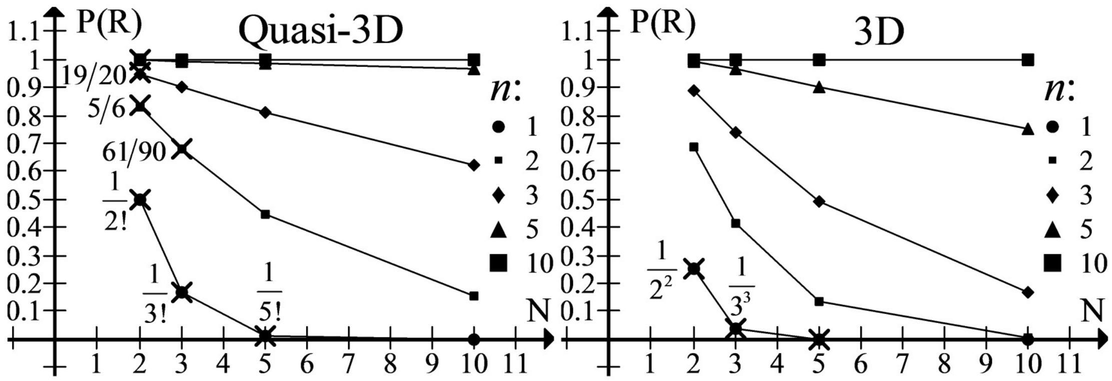

Fig. 8. Probability $P_R(N)$ of formation of a through inclusion when $r \geq R$ (one-dimensional case) as a function of N for n = 1; 2; 3; 5; 10 for the quasi-3D case on the left and for the 3D case on the right.

We also note that for one n (bottom pair of Fig. 7) graphs $P(r)$ in the region of rapid increase, i.e. near the inflection point of the S-shaped curve, run almost parallel to each other.

The same is observed for the layered case [15] with a uniform distribution of islands. However, with a non-uniform distribution, the graphs for different degrees of heterogeneity can intersect and overtake each other.

Same view of $P(r)$ and the same monotonic nature of displacements of $P(r)$ when N, n changes allow us to describe $P(r)$ by a small number of numeric parameters:

1. Probability $P_R = P(R)$ — right upper point of the S-shaped curve, one-dimensional limit, as described above.

2. Percolation threshold analogue for this case, value r, at which $P = \frac{1}{2}P(R)$ — position of the "center" of the S-curve $P(r)$. We will call this quantity the percolation radius and denote it r_c.

3. Slope t_c of S-shape $P(r)$ at an inflection point (near or coinciding with r_c). This value will be needed when studying the marked parallel course of $P(r)$ at one n. Also t_c characterizes proximity of $P(r)$ to the threshold view.

These three parameters define the S-curve quite fully and have a physical meaning.

Because probability P does not depend on r, but from r/R, we will further assume for simplicity $R = 1$.

First parameter, probability P(r).

For the quasi-three-dimensional case, the probability $P(R)$ was found above analytically for some n at $N = 2$ and some N at $n = 2$. For $n = 1$ coefficients were found to obtain $P(r)$ from $P^l(r)$ for the layered case for any r. Because for layered case $P(R)$, these coefficients are values of $P(R)$.

Values of $P(R)$ as dependencies $P_R(N)$ from the number of layers N for n = 1; 2; 3; 5; 10 found from the numerical experiment are shown in Fig. 8 for the quasi-3D case on the left and for the 3D case on the right. The crosses also show the values found above analytically (values 1506/1512 left and $1/5^5$ right are shown, but not signed).

Graphs $P_R(N)$ for small n have the form of a concave decreasing function, with increasing n approach the **linear form**, and at even greater n tend to constant $P(R)$=1. For a more detailed study, dependencies $P_R(N)$ were built in the range N = 2...100 for n= 3; 5, 7; 9, Fig. 9.

The resulting dependencies are very well approximated by exponential functions of the form

$$P_R(N) = (a(n))^{N-1} \tag{9}$$

(a different for quasi-three-dimensional and three-dimensional distributions).

Previously, we encountered a similar dependency P from N, (3). It means (if performed exactly) that the probability of forming a through inclusion is the product of the probabilities, which are the same, of the presence of a contact in each pair of layers. For this, two conditions must be met. First, to form a through inclusion, it should be sufficient to have a contact in each pair of layers. At $r < R$ this condition is met only when $n = 1$, because at $n > 1$ one island in the layer can have contact with the bottom layer, and the other with

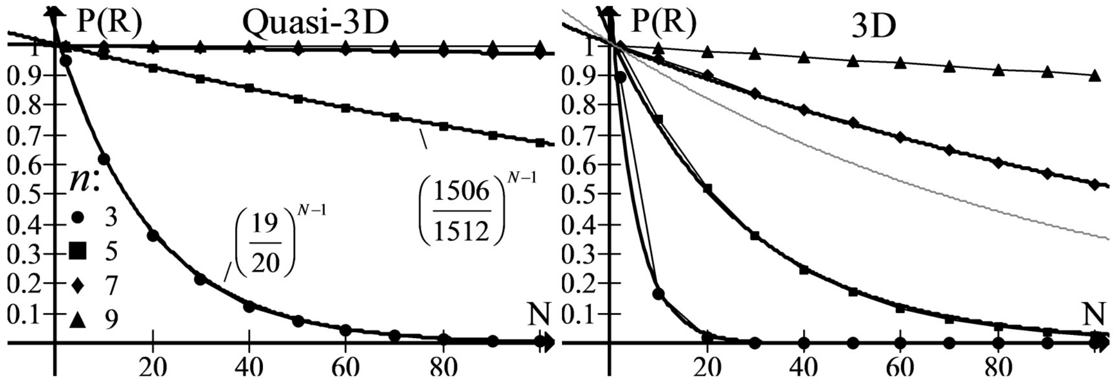

Fig. 9. Probability of forming a through inclusion at $r \geq R$ (one-dimensional case, the top point of the graph $P(r)$) as a function of N, for N up to 100, for $n = 3$; 5; 7; 9. For the quasi-three-dimensional case on the left and for the three-dimensional case on the right. The solid lines show the approximations $P_R(N) = (a(n))^{N-1}$.

the top layer, but not have contact with each other. I.e. there will be contact in each pair of layers, but there will be no through connection. The second condition is that the probability of contact between any two layers must be the same. In the general case, this is not the case, because to form a through inclusion, the probability of contact between islands in ν-th and ν+1-th layers should be determined under the condition of contact in the previous ν–1 pairs of layers, and it may depend (for example, for round layers) on ν.

At $r \geq R$ the first condition is met automatically, but the second condition, generally speaking, is not satisfied exactly. For example, expressions (7), (8) for the quasi-three-dimensional case do not break down into independent factors for each pair of layers. And in the three-dimensional case, the number of islands in the layers will be different, so that the pairs of layers are not equal in principle.

However, the fact that dependence (9) is a good approximation for $P_R(N)$, allows one to suggest that the condition is satisfied, at least approximately. In this case $a(n)$ in (9) must be equal to the value $P_R(2)$ for this n. For the quasi-three-dimensional case, the values $P_R(2)$ found above analytically. So for $n = 3$ should be $a(3) = 19/20$ and for $n = 5$ $a(5) = 1506/1512$.

Graphs of the functions $P_R(N) = (19/20)^{N-1}$ and $P_R(N) = (1506/1512)^{N-1}$ are shown in Fig. 9 left. Also graph of the function $P_R(N) = (a(7))^{N-1}$ is shown, where $a(7)$ equals to $P_R(2)$ for $n = 7$ taken from numerical experiment. They approximate the values of $P_R(N)$ very well for relevant n.

When increasing in n base $a(n)$ rapidly tends to unity, so that already at $n = 5$ until $N = 100$

$$P_R(N) = (1 - \varepsilon(n))^{N-1} \approx 1 - \varepsilon(n)(N-1) \quad (10)$$

runs, which explains the linear $P_R(N)$ form noted above at not small n.

For the three-dimensional case, Fig. 9 on the right, approximation (9) is also performed very well, the corresponding graphs are shown in Fig. However, $a(n)$ for the three-dimensional case **is not equal** to P_R at $N = 2$. For an example in Fig. the graph of function (9) is shown by a gray line at $a(n)$ equal to $P_R(2)$ for $n = 5$ from a numerical experiment. As you can see, it is very far from $P_R(N)$ for $n = 5$ — second graph from the bottom. This means that the condition of the independence of the contact in each pair of layers in the three-dimensional case is approximately satisfied for large N, but not at small.

Percolation radius r_c.

Let's start with the case $n = 1$. Because equality (3) is satisfied quite accurate, for the layered case r_c can be found analytically by equating the right side of (3) to 1/2.

For the quasi-three-dimensional and three-dimensional case, the probability $P(r)$ is equal to the right side of (3) divided by $N!$ and N^N respectively, and to find r_c this value should be equated to $P(R)/2$ for the case under consideration. But $P(R)$ is also equal to the ratio of $P(R)$ for layered case, unity, to $N!$ and N^N, so the equation for r_c for the considered cases at $n = 1$ the same

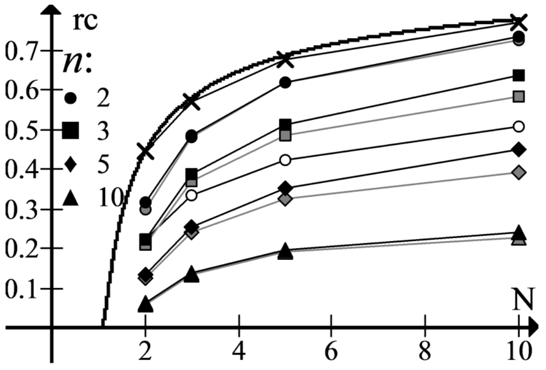

Fig. 10. Values r_c of percolation radius — values r, at which $P = 1/2\ P(R)$, as a function of N. For the quasi-3D case in gray, for the 3D case in black.

as for the layered, and hence the values r_c the same.

Graph of analytical $r_c(N)$ for $n = 1$ shown in Fig. 10 as a solid line. The crosses show the values r_c at $n = 1$ found from the numerical experiment (for the layered case). As can be seen, for $N = 2$ the resulting value lies strictly on the graph, as it should be, and for $N > 2$ the values differ slightly, because equality (3) is satisfied approximately.

Values r_c for $n > 1$ were determined from a numerical experiment. Values $r_c(N)$ shown in Fig. 10 for the quasi-3D (in grey) and 3D (in black) case for $n = 2;\ 3;\ 5;\ 10$.

As you can see, the values $r_c(N)$ for quasi-three-dimensional and three-dimensional cases for $N = 2...10$ close, for $n = 2$ almost identical, for $n = 3;\ 5$ almost coincide at $N = 2$ and diverge with increasing of N, for $n = 10$ approach again almost to coincidence for the considered N.

However, presented r_c do not match $r_c(N)$ for the layered case. For an example in Fig. 10 $r_c(N)$ shown for the layered case with $n = 2$, empty circles.

Rapprochement $r_c(N)$ for large $n = 10$ due to the effect described above — the distribution of islands in the three-dimensional case approaches the distribution for the quasi-three-dimensional case at large n. With an increase of N this effect is reduced.

Angular coefficient t_c at the inflection point of the S-shaped $P(r)$.

At $n = 1$ for quasi-3D and 3D case $P(r)$ differs from $P(r)$ for the layered case by a constant factor: $1/(N!)$ and $1/N^N$ respectively, while $P^l(r)$ changes from N exponentially (3). So the value t_c will change from N. On the other hand, for $n > 1$ it was above marked approximate parallelism of $P(r)$, i.e. approximate equality of t_c at one n for different N, see example on Fig. 7 above. The same parallelism was observed for the layered case.

To study this parallelism, graphs $t_c(N)$ were built at one n, Fig. 11 above. From left to right — for the layered case, quasi-three-dimensional and three-dimensional. As it seen, $t_c \neq const$, but for $n \sim 3...5$ changes are small. At large $n = 10$ as with $n = 1$, for non-layered cases t_c changes from N noticeably. For $n = 10$ changes are nonmonotonic.

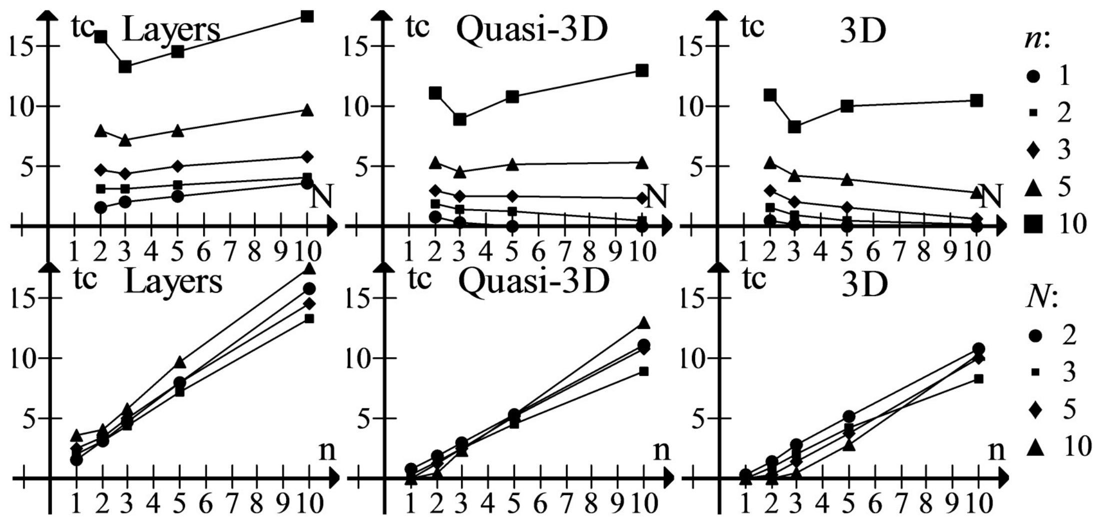

Fig. 11. Values t_c of slope of S-shaped $P(R)$ at the inflection point, as a function of N for a given n top, as a function of n for a given N bottom. For the layered case (left), quasi-3D (center), and 3D (right).

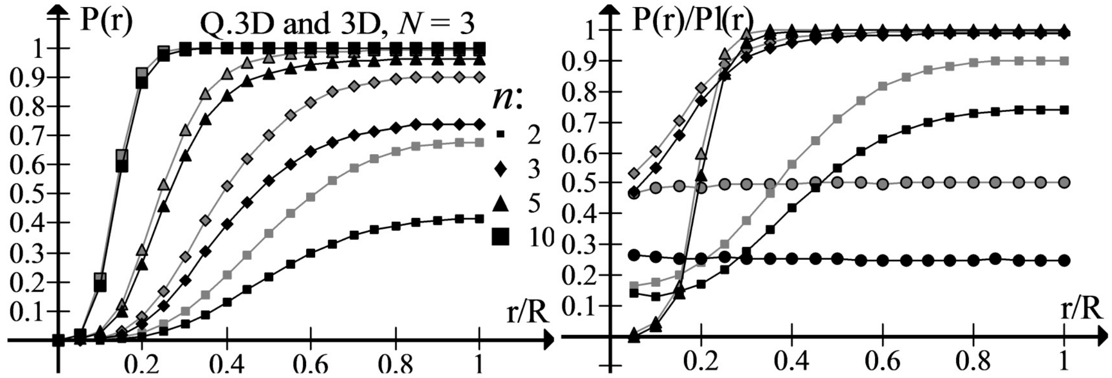

Fig. 12. Left — an illustration of the approximation of $P(r)$ for the three-dimensional case to $P(r)$ for quasi-3D with increasing of n. On the right — typical cases of the relationship $P(r)/P^l(r)$ as functions of r.

For research of t_c change with n graphs $t_c(n)$ at one N were built, Fig. 11 below. As can be seen, changes for small N are approximately linear. The same was observed for the layered case. Closest to linear changes at $N = 2$.

Values t_c for the quasi-three-dimensional and three-dimensional cases are close at $N = 2$; 3, but the differences increase with increasing of N. Rapprochement t_c with increasing of n and divergence with increasing of N is related to the approximation of the distribution of islands along the vertical for the three-dimensional case to the quasi-three-dimensional one with increasing of n.

Additional properties of P(r).

It was stated above that the values of P for the quasi-three-dimensional and three-dimensional cases should become close at large n. This property is illustrated by the results of a numerical experiment in Fig. 12 left. Graphs $P(r)$ shown for quasi-three-dimensional (gray) and three-dimensional (black) cases with one $N = 3$ for $n = 2$; 3; 5; 10. As can be seen, with increasing in n the graphs converge and at $n = 10$ almost identical. The effect decreases with increasing of N, i.e. to observe the same approach, you need at larger N take larger n. This is due to an increase in the influence of fluctuations in the number of islands on the layer for the three-dimensional case with an increase in the number of layers.

For $n = 1$ above, the values of the ratio $P(r)$ to $P^l(r)$ were found analytically, (2), (5) for the same N, r. These relationships do not depend on r. At $n > 1$ relation $P(r)$ to $P^l(r)$ depends on r.

Typical dependencies of such a relation on r shown in Fig. 12 on the right, in gray for the quasi-3D case and in black for the 3D case, for the same N, n. First type: $P(r)/P^l(r) = const$, marked with circles, corresponds to $n = 1$ (in Fig. $N = 2$, $n = 1$). The second type is marked with squares: $P(r)/P^l(r)$ varies in the interval $[\kappa_1, \kappa_2]$, $\kappa_1 > 0$, $\kappa_2 < 1$. It is typical for cases when at the same time N "not too large" for the given n and n "not too large" for the given N (in Fig. $N = 3$, $n = 3$). The third type, marked with diamonds: $P(r)/P^l(r)$ varies from values greater than zero to one. Typical for the case when n "large enough" for the given N (in Fig. $N = 2$, $n = 5$), so $P(r) \to 1$ at $r \to R$. Finally, the fourth type is typical for the case when N "large enough", then $P(r)/P^l(r)$ varies from zero or values close to zero. The combination of the third and fourth types are dependencies $P(r)/P^l(r)$, varying from zero to one, example ($N = 5$, $n = 10$) is marked in Fig. by triangles.

In Fig. 12 on the right, the rapprochement $P(r)/P^l(r)$ is also noticeable for quasi-three-dimensional and three-dimensional cases with increasing in n, almost to coincidence.

Conclusions

Values $P(r)$ received at various N, n analytically and numerically. Two cases of vertical distribution of inserts in the sample are considered — the case of displacements in the interval $\pm\Delta$ from the center of the layer in separate layers and the case of a uniform distribution of islands in the sample. For the case of displacements, it is shown that the probability $P(r)$ does not depend on the amplitude Δ, from arbitrarily small nonzero to $h/2$, quasi-three-dimensional case.

1. It is shown that at $n = 1$ $P(r)$ for the quasi-three-dimensional and three-dimensional cases can be obtained from $P^l(r)$ for the layered case by multiplying by a factor equal to $1/N!$ for quasi-3D and $1/N^N$ for the three-dimensional case and independent of r. Together with analytical expression (3) for $P^l(r)$ this allows one to get analytically $P(r)$ for the considered cases at $n = 1$. The results of the numerical experiment coincide with the analytical ones, Fig. 3.

2. It is shown that at $r \geq R$ the problem becomes one-dimensional, herewith the probability P for all $r \geq R$ is the same. For layered case $P(R)$ is strictly equal to one. For the quasi-three-dimensional case, expressions for $P(R)$ for $N = 2$ and any n obtained explicitly (7) and for $n = 2$ and any N in integrals (8). The results of the numerical experiment coincide with the analytical ones, Fig. 6.

3. For the general case of arbitrary N, n it is shown, that $P(r)$ have the same S-shape for any N, n. And in the series of $P(r)$ at constant N and various n or constant n and various N, graphs are arranged without intersections, strictly from left to right when increasing in N and from right to left when increasing in n. That's why $P(r)$ with given N, n is quite fully described by several numerical parameters of the S-curve: $P(R)$ — top point of the curve, "percolation radius" r_c — position of the center of the curve, slope t_c at the inflection point of the S-curve $P(r)$.

4. It is shown that the values $P(R)$ as functions of N both for the quasi-three-dimensional and for the three-dimensional case are well approximated by functions of the form $P_R(N) = (a(n))^{N-1}$ (9), Fig. 8, 9. For the quasi-three-dimensional case $a(n) = P_R(2)$, which means approximately independent contact formation in each pair of layers at $r \geq R$. With an increase in n $P_R(2)$ quickly tends to unity, which leads to an almost linear relationship $P_R(N)$ (10) even for sufficiently large N. For the 3D case $a(n) \neq \Pi_P(2)$, ι.ε. χονταχτ ινδεπενδενχε ισ οβ–σερϖεδ ατ λαργε Ν.

5. Dependencies $r_c(N)$ as functions of N have the same form for the layered, quasi-three-dimensional and three-dimensional cases for any n, Fig. 10 — function increasing with deceleration. Values r_c for the quasi-three-dimensional and three-dimensional cases are close, but noticeably differ from r_c for the layered case with the same N, n. $r_c(N)$ almost match for $n = 2$.

6. Slope t_c of graphs $P(r)$ at the inflection point for small $n > 1$, $n \sim 3...5$, approximately the same at the same n and does not depend on N, Fig. 7, 11 above. With one N, t_c changes from n almost linear, Fig. 11 below. At large $n = 10$ t_c depends on N. At $n = 1$, from the solutions found analytically, t_c changes from N.

For large n, r_c and t_c converge for the quasi-three-dimensional and three-dimensional case with increasing of n and diverge with increasing of N.

7. Numerical experiment confirms the approximation of $P(r)$ for the three-dimensional case to $P(r)$ for quasi-three-dimensional at large n, Fig. 12 left. This effect is due to the fact that the distribution of islands along the sample vertical in the three-dimensional case approaches the quasi-three-dimensional one with a relative deviation $1/\sqrt{(n)}$. The effect decreases with increasing of N.

8. Analytically found relation $P(r)/P^l(r)$ for $n = 1$ does not depends on r, however at $n > 1$ it depends on r and can change as in the interval $[\kappa_1, \kappa_2]$, $\kappa_1 > 0$, $\kappa_2 < 1$ and reach the values 0 and/or 1 depending on N, n, Fig. 12 right.

References

1. I.F.Mello, L.Squillante, G.O.Gomes et al., *Physica A: Statistical Mechanics and its Applications*, **573**, 125963 (2021).
2. R.M.Ziff, *Physica A: Statistical Mechanics and its Applications*, **568**, 125723 (2021).
3. L.Vassallo, M.A.Di Muro, D.Sarkar et al., *Phys.Rev.E*, **101**, 052309 (2020).
4. D.Duffie, G.Giroux, G.Manso, *American Economic Journal: Microeconomics*, **2**, 100 (2010).
5. D.Duffie, S.Malamud, G.Manso, *Journal of Economic Theory*, **153**, 1 (2014).
6. F.Bagnoli, E.Bellini, E.Massaro, R.Rechtman, *Future Internet*, **11**, 35 (2019).
7. J.Xie, F.Meng, J.Sun et al., *Nat Hum Behav*, **5**, 1161 (2021).
8. E.Hyytia, J.Virtamo, P.Lassila, J.Ott, *IEEE Communications Letters*, **16**, 1064 (2012).
9. N.K.Shrivastava, B.B.Khatua, *Carbon*, **49**, 4571 (2011).
10. J-u Jang, H.E.Nam, S.O.So et al., *Polymers*, **14**, 323 (2022).
11. A.Motaghi, A.Hrymak, G.H.Motlagh, *J.Appl. Polym.Sci.*, **132**, 41744 (2015).
12. B.I.Armitage, K.Murugappan, M.J.Lefferts et al., *J.Mater.Chem.C*, **8**, 12669 (2020).
13. T.Sauerwald, S.Russ, Gas Sensing Fundamentals. Springer Series on Chemical Sensors and Biosensors, **15**, 247 (2013).
14. K.Sandberg, J.G.Salin, *Wood Sci Technol.*, **46**, 207 (2012).

15. R.Ye.Brodskii, *Funct. Mater.*, **27**, 159 (2019).
16. J.Dominique, M.Moreaud, *Image Analysis & Stereology*, **26**, 121 (2007).
17. S.I.White, R.M.Mutiso, P.M.Vora et al., *Advanced Functional Materials*, **20**, 2709 (2010).
18. J.Tykesson, D.Windisch, *Probab. Theory Relat. Fields*, **154**, 165 (2012).
19. M.R.Hilario, V.Sidoravicius, A.Teixeira, *Probab. Theory Relat. Fields*, **163**, 613 (2015).
20. R.Rossignol, M.Theret, *Stochastic Processes and their Applications*, **120**, 873 (2010).
21. M.G.Kendall, P.A.P.Moran, *Geometrical Probability, C. Griffin (1963).*
22. R.Ye.Brodskii, T.V.Kulik, *Funct. Mater.*, **29**, 419 (2022).
23. R.Ye.Brodskii, T.V.Kulik, *Funct. Mater.*, **29**, 586 (2022).