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Formation of "windows" of transparency in a layered samplewith an inhomogeneous distribution of inclusions in the layers; special case

R.Ye.Brodskii

Institute for Single Crystals, STC "Institute for Single Crystals",
National Academy of Sciences of Ukraine,
60 Lenin Ave., 61001 Kharkiv, Ukraine
V.Karazin National University, sqr. Svobody 4, Kharkiv, 61022, Ukraine

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In this work we studied the formation of transparency "windows" in a layered sample with the layers consisting of a basic, opaque material with transparent inclusions, in the case when islands of the transparent material are distributed inhomogeneously in the layer. The distribution densities of the "windows" over the area were obtained for inhomogeneities of various types and different values of "thickening" of the islands toward the center or edge of the layer. The special solutions for the case of a single island in a layer in a two-layer sample are considered.
Keywords: "windows" of transparency, layered sample, transparent islands

Формування вікон прозорості у шаруватому зразку при неоднорідному розподілі включень у шарах. Спеціальний випадок. *Р.Е.Бродський.*
Вивчено формування вікон прозорості у шаруватому зразку, шари якого складаються з основного, непрозорого матеріалу з включеннями прозорого, в разі, коли острівці прозорого матеріалу розподілені неоднорідно у шарі. Отримані густини розподілу "вікон" за площею для неоднорідності різного виду і різної величини "згущення" острівців до центру або краю шару. Досліджено спеціальний випадок єдиного острівця у шарі у зразку з двох шарів, що має особливі рішення.

Изучено формирование окон прозрачности в слоистом образце, слои которого состоят из основного, непрозрачного материала с включениями прозрачного, в случае, когда островки прозрачного материала распределены неоднородно в слое. Получены плотности распределения "окон" по площади для неоднородности различного вида и различной величины "сгущения" островков к центру или краю слоя. Исследован специальный случай единственного островка в слое в образце из двух слоёв, имеющий особые решения.

1. Introduction

When individual layers in a layered sample can contain, in addition to the main material, inclusions of additional material with properties different from the main one, such inclusions can contact with each other in the layers; in this case, bulk inclusions, including penetrating through the sample, can occur. These bulk inclusions can qualitatively change the properties of the sample as a whole. An example is conductive inclusions in a non-conductive material of layers or transparent inclusions in an opaque material.

In the case when the transparent inclusions are strictly one above the other in all opaque layers in some region of the sample, a through "window" of transparency appears in the sample, from the first to the last layer. Due to the intersection and over-

lap of transparent regions on top of each other within the layer, the resulting "windows" can have a very varied complex shape and a very varied area. In the task of forming "windows" of transparency, the goal is to obtain the corresponding distribution of "windows" over the area for the given parameters of the formation of the sample and inclusions in separate layers. The formation of transparency "windows" or regions of conductivity in a sample depends on the number of layers, the number and size of inclusion islands in individual layers, and the type of distribution of these islands in the layer.

The case of distribution of islands (centers of circular islands) that are homogeneous and identical in all layers is considered in [1] for the problem of transparency "windows" and in [2] for the problem of conductivity. Cases when the formation of islands in a given layer depends on their distribution in the previous layer are considered in [3–5] for the transparency "windows" problem. The sample was considered to be grown layer by layer, from "bottom" to "top". In [3], the case was considered when islands were nucleated (that is, their centers were located) only above the material of the same, transparent, "phase"; in [4–5], the case was discussed when islands were nucleated only above the "phase boundary". Such problems for the case of conductivity do not need to be considered, since with this type of inclusions, a through inclusion is always formed in the layers (if there are any conductive inclusions in the "upper" layer). It was shown in [1] that the case of two layers with one island in each has special solutions that differ significantly from the solutions for an arbitrary number of layers and islands in the layer. Moreover, in this case the solutions were obtained analytically [1]. In this paper, we will consider just such a special case.

In this paper, we consider the case of the formation of inclusions which are independent and equally *inhomogeneous* in the area of the layer. The formation of islands in each layer does not depend on the distribution of the islands in the adjacent layer or on the layer number. The probability of nucleation of an island in a given element of the layer region (i.e., the probability of the location of the center of the island) depends on the position (coordinates) of the element in the layer. The purpose of the work is to study the effect of various types and values of such inhomogeneity on the formation of transparency "windows".

2. Formulation of the problem

We consider a cylindrical sample consisting of $N = 2$ round layers with a radius R. Each layer consists of a basic opaque material and transparent inclusion. An inclusion is the circular island with a radius r. The layers are considered thin and "flat", i.e. the edges of all inclusions, like the edge of the layer, are perpendicular to the plane of the layer.

In this paper, we will consider the case of an inhomogeneous distribution of islands (their centers) in the layer in a comparison with the "homogeneous" case; also, the change in the distribution of the "windows" over the area with an increase in the degree of inhomogeneity will be studied.

The distribution of islands in the layer is determined by the density $f(x,y)$ of distribution of their centers over the area of the layer. In the case of uniform distribution, $f(x,y) = const$. The layers of the sample have the shape of circles, so we consider centrally symmetric densities, which depend only on the distance to the center of the layer. It is convenient to use scale-invariant quantities. We introduce ρ as the distance to the center of the layer, related to the radius of the layer R.

We introduce the density of distribution of islands in the layer $f(\rho)$, such that $f(\rho)\cdot d\rho\cdot\rho d\varphi$ is the probability that the island originates (i.e., its center will be located) in the element $(\rho, \rho + d\rho)\times(\varphi, \varphi + d\varphi)$ of layer ($f(\rho)$ is independent of φ). The function $f(\rho)$ is normalized to one,

$$\int_0^1 f(\rho)\cdot 2\pi\rho d\rho = 1.$$

In particular, with a uniform distribution of islands: $f(\rho) = f_0 = 1/\pi$.

A decrease in the function $f(\rho)$ corresponds to a greater probability (per unit area) of the nucleation of islands in the center of the layer, we will call it "center"; an increase in the function $f(\rho)$ with ρ corresponds to a greater probability of the nucleation of islands at the edge, such $f(\rho)$ we will call "edge".

It is convenient to use such density functions $f(\rho)$ which make it possible to obtain the entire spectrum of "degree of inhomogeneity" for various values of the control parameter — from the homogeneous case $f(\rho) = const$ to an arbitrarily strong thick-

ening of the islands in the center or at the edge of the layer.

For the "central" case, this condition corresponds to the Gaussian $f(\rho)$,

$$f(\rho) = A\exp(-\alpha\rho^2).$$

Coefficients A, α are related by the normalization condition, so that the function has one control parameter.

When $\alpha = 0$, the homogeneous case is realized; with increasing α you can get as sharp as you need the probability "peak" in the center of the layer.

For the "edge" case, the specified condition corresponds to a power $f(\rho)$ function.

$$f(\rho) = A\rho^{\alpha}.$$

As for a Gaussian function, the coefficients of this function are related by the normalization condition, the function has one control parameter. At $\alpha = 0$ the homogeneous case is realized, with increasing α you can get an arbitrarily narrow "ring" of the probable position of the centers of the islands at the edge of the layer.

As a numerical characteristic of the "degree of inhomogeneity" we will use the distance ρ_w to the center of the layer, such that the probability of the nucleation of an island "inside" it ($\rho < \rho_w$) for a given $f(\rho)$ is equal to 0.5 (and "out" it, respectively, the same amount). Thus, this is the characteristic of width (hence, the ρ_w "width") of the curve of the probability distribution (not density) of nucleation over ρ. The stronger the "central" inhomogeneity, the less ρ_w the stronger the "edge" inhomogeneity, the closer the ρ_w to one. For the homogeneous case, the ρ_w value is denoted ρ_{w0}; we have

$$\int_0^{\rho_{w0}} \frac{1}{\pi} \cdot 2\pi\rho d\rho = 0.5,$$

from where $\rho_{w0} = 1/\sqrt{(2)}$. For the "central" inhomogeneity: $\rho_w < \rho_{w0}$, for the "edge" inhomogeneity: $\rho_w > \rho_{w0}$.

To compare the behavior of the system with the same ρ_w but different $f(\rho)$ we additionally consider the linear function $f(\rho) = A \pm \alpha\rho$. Here, $A > 0$ and $\alpha \geq 0$, the coefficients are related by the normalization condition, so that there is a single control parameter. The option with "–" corresponds to "central", and with "+" — to "edge" inhomogeneity.

Note that possible ρ_w for linear $f(\rho)$ is limited because the possible values of function parameters are limited by the condition $f(\rho) \geq 0$ at $\rho \in [0, 1]$ that for the two functions above is performed automatically.

For the "central" ("–") linear $f(\rho)$ function, this condition matters on the edge, $f(1) \geq 0$. The edge case $f(1) = 0$ corresponds to $A = \alpha$ that, taking into account the normalization, gives $A = \alpha = 3/\pi$ and $(\rho_w^2 - 2/3\rho_w^3) = 1/6$, from where $\rho_w = 0.5$. This is the lowest possible value ρ_w for the "center" linear $f(\rho)$.

Similarly, for the "edge" ("+") linear $f(\rho)$, this condition matters in the center, $f(0) \geq 0$; the limiting case $f(0) = 0$ corresponds $A = 0$ that, taking into account the normalization, gives $\alpha = 3/2\pi$ and $\rho_w = 1/(2)^{1/3} \approx 0.7939$. This is the maximum possible value ρ_w for the linear $f(\rho)$ function.

Thus, we will carry out a comparison with the previous ones, Gaussian and power $f(\rho)$ functions only in the range of attainable values ρ_w for a linear function.

Note one more thing. Although the power function transforms into a constant (homogeneous case) with $\alpha = 0$, for an arbitrarily small non-zero α, this function has a dip in the area of $\rho = 0$, because $f(0) = 0$ for any $\alpha \neq 0$ (positive). So, it is more convenient to study the "edge" case in the field of ρ_w close to ρ_{w0}, not with power, but with another, a linear $f(\rho)$ function, while the power $f(\rho)$ function will be used only when α is sufficiently far from zero, i.e. with a sufficiently strong inhomogeneity.

Although the Gaussian $f(\rho)$ function does not have such features, we will use the linear also to study the "central" inhomogeneity in the region ρ_w close to ρ_{w0}, while the Gaussian will be used for a sufficiently strong "central" inhomogeneity.

We concretize the concept of "sufficiently strong inhomogeneity". As shown above, the minimum ρ_w (the strongest inhomogeneity) for the "central" linear inhomogeneity is 0.5. We use a Gaussian $f(\rho)$ for ρ_w equal to this value and smaller ones. Namely, for $\rho_w = 0.1$, $\rho_w = 0.3$ and $\rho_w = 0.5$.

For the "edge" case, we introduce ρ_w corresponding to a "sufficiently strong" inhomogeneity, in which we use the power law of $f(\rho)$ in the following way. The selected values ρ_w for the "strong central inhomogeneity" divide the range of values $\rho_w(0, \rho_{w0})$ corresponding to the "central" inhomo-

geneity, in some proportion. For the "strong edge inhomogeneity" we choose ρ_w values which divide the interval (1, ρ_{w0}) (from larger to smaller) corresponding to the "edge" inhomogeneity, in the same proportion. The value $\rho_w = 0.1$ will match the value $\rho_w = 1 - 0.1/\rho_{w0} \cdot (1 - \rho_{w0}) \approx 0.96$, and similarly, $\rho_w = 0.3$ will match $\rho_w = 0.876$, and value $\rho_w = 0.5$ will match $\rho_w = 0.7929$. The last value almost coincides with the maximum ρ_w for linear $f(\rho)$ that allows us to compare the results for linear and power $f(\rho)$ functions.

Another function $f(\rho)$ used in the work will be introduced in the next section.

3. A special case, the only island in the layer

It was previously shown, that for a uniform distribution of islands in the layer, for the case of a single islandin the layer both the distribution density of "windows" over the area [1] and the probability of the formation of a through conductive inclusion [2] one can obtain analytically.

The case of the only island in the layer is special. We can explicitly write an expression for the area of the "window" (the "window" has a lenticular shape) in the case of two layers; in the case of many layers, if the "window" exists, there is only one. In this selected case, the problem is reduced to finding the density $p(x)$ of distribution of distances between the centers of islands in neighboring layers, i.e., formally, between two random points in a circle. In the homogeneous case, these two random points are considered uniformly distributed; now consider random points distributed with a density $f(\rho)$.

If this distance density $p(x)$ is known, then in the case of two layers, the expression for the distribution density $p_s(S)$ of "windows" of transparency over the area can be written exactly (implicitly)

$$\begin{cases} p_s(S) = p(x)\dfrac{1}{dS/dx}, & x \le 2r, \\ S = S(x) \end{cases} \qquad (1)$$

Here $S(x)$ is the expression for the area of the "window" formed by two superimposed islands, one in each layer; x is the distance between the centers of the islands. The case is considered when the "window" lies completely inside the layer and its part is not cut off by the boundary of the layer.

In the homogeneous case the $p(x)$ function can be obtained analytically in an explicit form. We try to obtain the $p(x)$ analytically, at least for some types of inhomogeneity. For those types of inhomogeneity for which it is impossible to obtain the $p(x)$ function analytically, we will obtain the values $p(x)$ numerically and use them to determine the distribution density of the transparency "windows" over the area, rather than by direct numerical simulation. Firstly, this is convenient for comparing with the results obtained in the homogeneous case; secondly, the form of $p(x)$ function by itself is a material for analyzing the effect of inhomogeneity on the formation of inclusions in layered samples.

In the homogeneous case, the density $p(x)$ could be found using Crofton's theorem [6], in which the problem of two random points in a circle is reduced to the much simpler problem of one point in a circle and another on the boundary. In the inhomogeneous case, we have to solve the problem of finding $p(x)$ directly.

Let one of the points be randomly distributed in the circle with density $f(\rho)$. We find the probability that the second same distributed point is located at a distance from the first one in the interval $(x, x + dx)$. This arrangement of the second point is shown in Fig. 1 (bold line) for $x > R$, for $\{x < R;\ x < R(1 - \rho)\}$ and for $\{x < R;\ x > R(1 - \rho)\}$ (from left to right). In the second case, the region where the second point can be located is a thin ring; in the first and the last cases, it is a thin arc layer.

The distance from the first point to the center of the layer is ρR, from the second — $\tilde{\rho}R$. Thus, the probability that the second point is at a distance $(x, x + dx)$ from the first is

$$\int_0^{2\pi} f(\tilde{\rho}(\rho,x,\psi)) \cdot dx \cdot x d\psi \quad \text{for}$$

$\{x<R;\ x < R\,(1 - \rho)\}$ (second case) and

$$\int_{-\theta}^{\theta} f(\tilde{\rho}(\rho,x,\psi)) \cdot dx\ x d\psi \text{ in the other two cases.}$$

Integration is performed by ψ. Here, the value θ is the maximum value for ψ (modulo),

$$\theta = \arccos \frac{R^2 - (\rho R)^2 - x^2}{-2 \cdot \rho R \cdot x}.$$

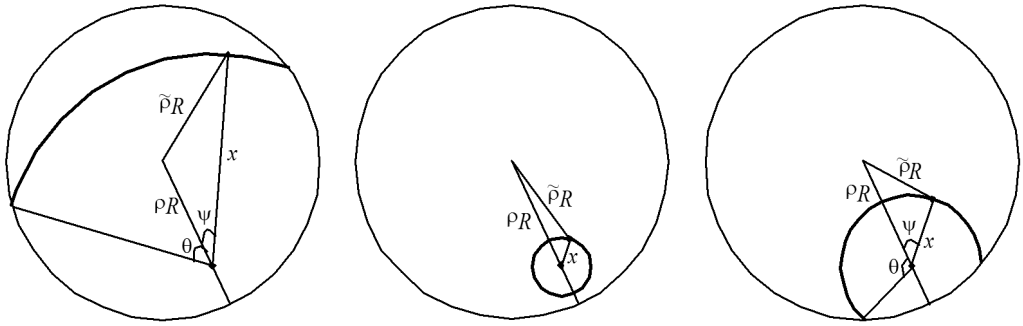

Fig. 1. Possible position of the second random point relative to the first at a distance $(x, x + dx)$, left to right the cases: $x > R$, $\{x < R, x < R(1 - \rho)\}$ and $\{x < R, x > R(1 - \rho)\}$.

Adding integration over the position of the first random point, we obtain for the distance density $p(x)$ between the points

$$p(x) = \int_{(x-R)/R}^{1} f(\rho) \cdot \left(\int_{-\theta}^{\theta} f(\tilde{\rho}(rho,x,\psi)) x d\psi \right) \cdot 2\pi\rho d\rho \quad \text{for } x>R \tag{2a}$$

(and, of course $x \le 2R$, this is the maximum possible distance between points in the circle) and

$$p(x) = \int_{0}^{(R-x)/R} f(\rho) \cdot \left(\int_{0}^{2\pi} f(\tilde{\rho}(\rho,x,\psi)) x d\psi \right) \cdot 2\pi\rho d\rho + \int_{(R-x)/R}^{1} f(\rho) \cdot \left(\int_{-\theta}^{\theta} f(\tilde{\rho}(\rho,x,\psi)) x d\psi \right) 2\pi\rho d\rho \quad \text{for } x \le R. \tag{2b}$$

From Fig. 1, it's clear that

$$\tilde{\rho}^2 = (x/R)^2 + \rho^2 - 2(x/R)\rho\cos\psi.$$

So, for analytical research, the density function $f(\rho)$ of a parabolic form is the most convenient

$$f(\rho) = A \pm \alpha\rho^2.$$

This is another case of $f(\rho)$ which we mentioned at the end of the previous section, that we use in the work.

Consider, for example, the "central" parabolic density $f(\rho) = A - \alpha\rho^2$. At $\alpha = 0$ it goes into a constant — density $f_0 = 1/\pi$ for the homogeneous case. From top, the coefficient α is limited by condition $f(1) \ge 0$ (from the general condition $f(\rho) \ge 0$).

To consider the degree of deviation of the $f(\rho)$ function from the homogeneous case, it is convenient to characterize it by the value of the "height" h of the function at the layer boundary, $h = f(1)/f_0$, $h \in [0, 1]$. The case $h = 1$ corresponds to a uniform distribution, $h = 0$ — the maximum inhomogeneity.

Let us try to obtain analytically (in an explicit form) an expression for $p(x)$ with this $f(\rho)$ from formulas (2) above. In addition, we perform numerical integration using these formulas with the same $f(\rho)$.

It is not possible to obtain an analytical form of $p(x)$ for other $f(\rho)$ listed above; for these cases we will receive $p(x)$ from a numerical simulation of the random placement of two points in a circle. This completely different method for obtaining $p(x)$ does not use of formulas (2). So, for verification, we can compare the results of these two approaches using the $f(\rho)$ functions obtained analytically.

Taking into account the normalization for quadratic $f(\rho)$ we have $A = 1/\pi + \alpha/2$ and expressing α through h, we get $f(\rho) = (1/\pi)(2 - h) - (2/\pi)(1 - h)\rho^2$.

For simplicity, we consider only the case $x > R$ (formula (2a)). For the selected $f(\rho)$, the density $p(x)$ for $x > R$ is expressed as

$$p(x) = \int_{(x-R)/R}^{1} f(\rho) \cdot x \left(\int_{-\theta}^{\theta} \left(\frac{1}{\pi}(2 - h) - \frac{2}{\pi}(1 - h)((x/R)^2 + \rho^2 - 2(x/R)\rho\cos\psi) \right) d\psi \right) \cdot 2\pi\rho d\rho$$

at $\theta = \arccos\dfrac{R^2 - (\rho R)^2 - x^2}{-2 \cdot \rho R \cdot x}$.

The integration for $h = 3/4$ gives the expression for $p(x)$ explicitly. This expression is very cumbersome (it takes about a page) and we do not present it; the graph built using this analytical expression is shown in Fig. 2. This analytical result can be compared with the results of numerical integration and numerical modeling. Fig. 2 shows a graph for $p(x)$ with the "center" parabolic inhomogeneity

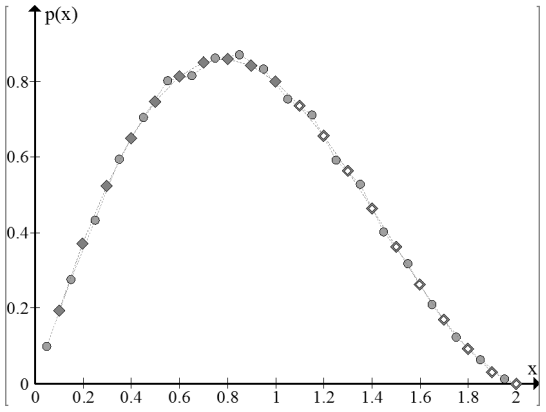

Fig. 2. Density of distances $p(x)$ at $f(\rho) = A - \alpha\rho^2$, $h = 3/4$, the circles show the results of modeling, gray rhombs — the numerical integration of analytical expressions, white small rhombs — the results of substituting into the expression obtained in an explicit form.

$f(\rho) = A - \alpha\rho^2$ with the value of the control parameter $h = 3/4$. In Fig. 2, circles show the simulation results, gray rhombs show the results of the numerical integration of the expression for $p(x)$ (for all x as larger than R as smaller ones) and white small rhombuses show the results of substituting into the expression obtained in an explicit form (for $x > R$).

As can be seen, the results of substitution into the formula and the results of numerical integration are the same, which confirms the correctness of the calculations. And both of these results are in very good agreement with the results of numerical modeling.

Fig. 3–8 show plots of density $p(x)$ of the distances between two random points in the circle and derived distribution densities $p_s(S)$ of "windows" transparency over the area for the case of a single island in the layer with $N = 2$ layers. This is a special case with a homogeneous distribution of islands, for which the distribution densities $p_s(S)$ of "windows" are obtained analytically in an explicit form [1]. It is seen that in the homogeneous case, these densities differ qualitatively in the cases of different island radii: $r < R$, $r > R$, and in the separate case $r = R$. At $r < R$, the density $p_s(S)$ tends to infinity with $S \to 0$; in this region, the $p_s(S)$ curve has a power asymptotics with a negative exponent. At $r > R$, the density $p_s(S)$ is nonzero only starting from some S_0, because the centers of the islands cannot be separated by a distance more than $2R$, and near $S = S_0$, the density plot has a power asymptotics with a positive exponent, $p_s(S_0) = 0$, the exponent is more than one. Finally, with $r = R$, the density plot $p_s(S)$ shows a power asymptotic for $S \to 0$ with a positive exponent, however, less than one.

We note that the analytical results for this special case in [1] were obtained under the assumption that the "window" formed by the superposition of the islands does not cross the layer boundary (which would make the analytical result too complicated and not so useful for analysis). In this case, we assume that R is the radius of the circular region in which the centers of the islands can be located (i.e., where the islands can grow from one starting point — their center — during layer formation). The radius of the layer is larger than R and sufficient so that the "window" does not intersect with the boundary of the layer. Here we will start from the same point of view.

Figs. 3–8 show graphs of $p(x)$ (top left, marked with the letter "a") and graphs of $p_s(S)$ for three cases: $r < R$, $r = R$ and $r > R$ (upper right, "b", left lower, "c" and lower right, "d", respectively). For the first case, we chose $r = 0.5R$ and for the latter case, $r = 1.2R$. For convenience, $R = 1$ was selected for all graphs.

Figs. 3–5 show $p(x)$ graphs for inhomogeneities of the "central" type for parabolic $f(\rho) = A - \alpha\rho^2$, linear $f(\rho) = A - \alpha\rho$ and Gaussian $f(\rho) = Ae^{-\alpha\rho^2}$ density functions, respectively. Figs. 6–8 show the graphs for inhomogeneities of the "edge" type: parabolic $f(\rho) = A + \alpha\rho^2$, linear $f(\rho) = A + \alpha\rho$ and power $f(\rho) = A\rho^\alpha$ density functions, respectively.

For parabolic and linear density functions $f(\rho)$, we choose a characteristic control parameter of inhomogeneity — the previously entered "height" h of functions $f(\rho)$ at the boundary of the layer (at $\rho = 1$, for the "central" inhomogeneity) or in the center of the layer (at $\rho = 0$, for the "edge" inhomogeneity) as more convenient in this particular case. For Gaussian and power $f(\rho)$ functions we used a more general characteristic of inhomogeneity ρ_w as a control parameter, which was also introduced above. To compare graphs $p_s(S)$ at different $f(\rho)$ dependences but with the same ρ_w values, we used both h and ρ_w parameters for parabolic or linear $f(\rho)$ functions. In each figure, the solid gray line shows the analytically obtained graph for the case of a

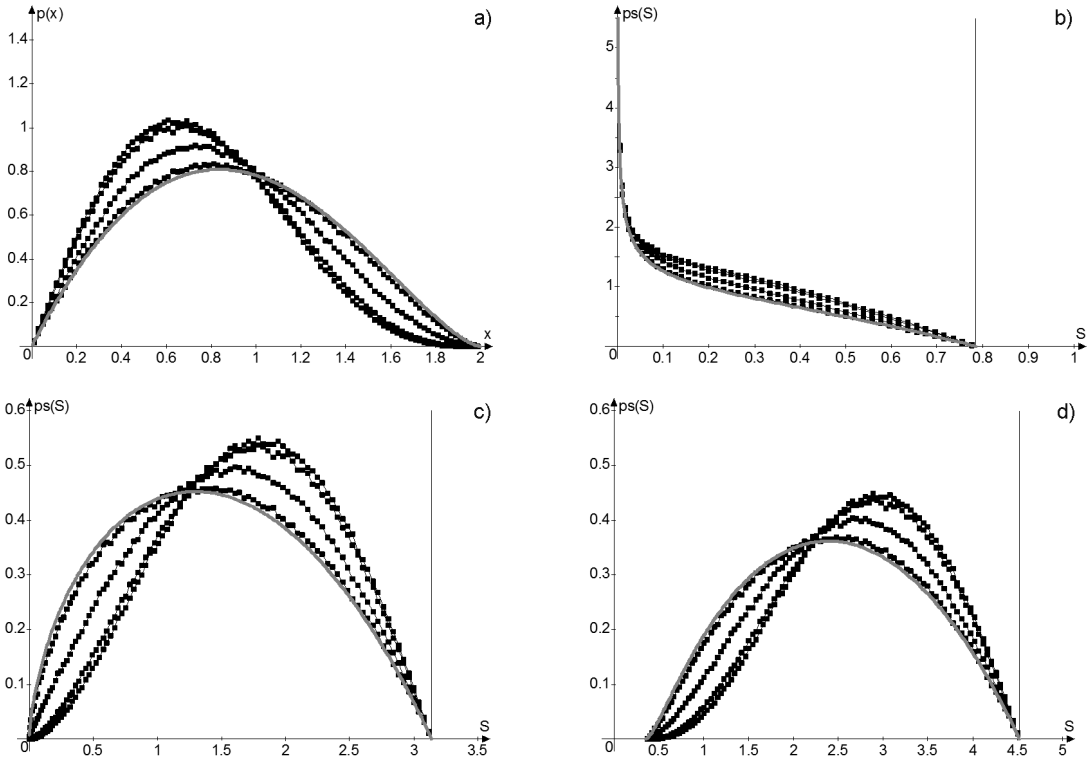

Fig. 3. The case of "central" parabolic inhomogeneity, $f(\rho) = A - \alpha\rho^2$, $h = 0$; 0.1; 0.5; 0.9.

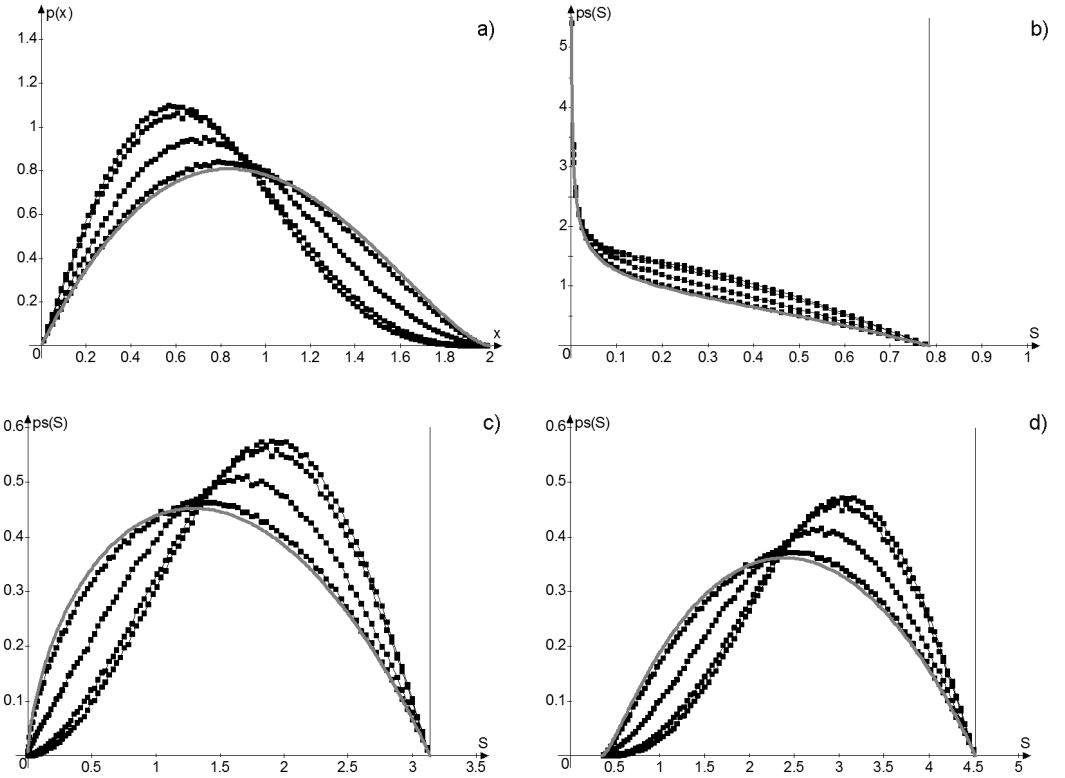

Fig. 4. The case of "center" linear inhomogeneity, $f(\rho) = A - \alpha\rho$, $h = 0$; 0.1; 0.5; 0.9.

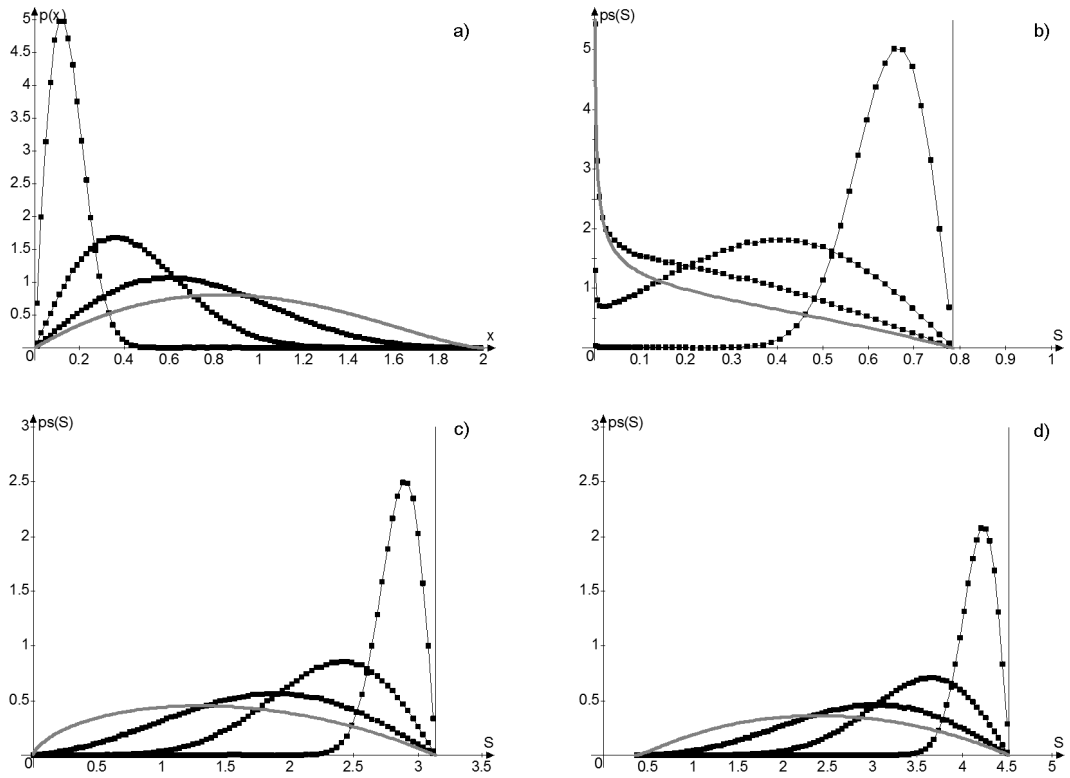

Fig. 5. The case of "central" Gaussian inhomogeneity, $f(\rho) = Ae^{-\alpha\rho 2}$, ρ_w = 0.1; 0.3; 0.5.

uniform distribution of islands with the same parameters.

In each picture of Fig. 3, the graphs from the bottom (closest to the graph shown by the gray line for a homogeneous case) to the top correspond to the change from the largest to smallest h values. Let us remind that $h = 1$ corresponds to a homogeneous case. Graphs for $h = 0.9$ almost coincide with the homogeneous case in each picture. Graphs for $h = 0$ and $h = 0.1$ almost coincide with each other. This means that small changes in the degree of inhomogeneity do not affect the appearance of the graphs, i.e. the system does not show a strong dependence on h.

As can be seen from Fig. 3a, the maximum of the $p(x)$ function shifts to the left with an increase in the degree of inhomogeneity; this expresses a decrease in the distances between random points, centers of islands, with an increase in the "central" (constricting islands to the center) inhomogeneity, as one would expect. The area of the formed "windows" should increase in this case, which we observe in Fig. 3b–3d — the graphs are shifted to the right, towards larger S.

The main qualitative change that we can note is the change in small-scale asymptotics in the special case $r = R$. As mentioned above, in this case, with a uniform distribution of islands, the asymptotic behavior for $S \to 0$ corresponds to a power law with an exponent less than unity (convex), while for $r > R$ the asymptotic exponent is greater than unity. From Fig. 3c, it can be seen that with increasing inhomogeneity, the graph for $r = R$ near $S = 0$ becomes concave, i.e. the qualitative difference from the case $r > R$ disappears.

This phenomenon can be commented as follows. With increasing central inhomogeneity, the "effective radius" of the distribution of island centers decreases, so that for the same radius of the island r, the condition $r = R$ is broken and the system goes into a state $r > R$.

The changes in the graphs with increasing degree of inhomogeneity for the "central" linear $f(\rho)$ function (Fig. 4) are the same as for the parabolic one. It can be noted that these changes are stronger than in the previous case; this is due to the fact that at the same h, the inhomogeneity with the linear "central" $f(\rho)$ distribution is stronger than with the parabolic function. On the graphs of $p_s(S)$ for the case of $r < R$, the convex region becomes visible; we will return to this effect when discussing the Gaussian inhomogeneity (Fig. 5). As men-

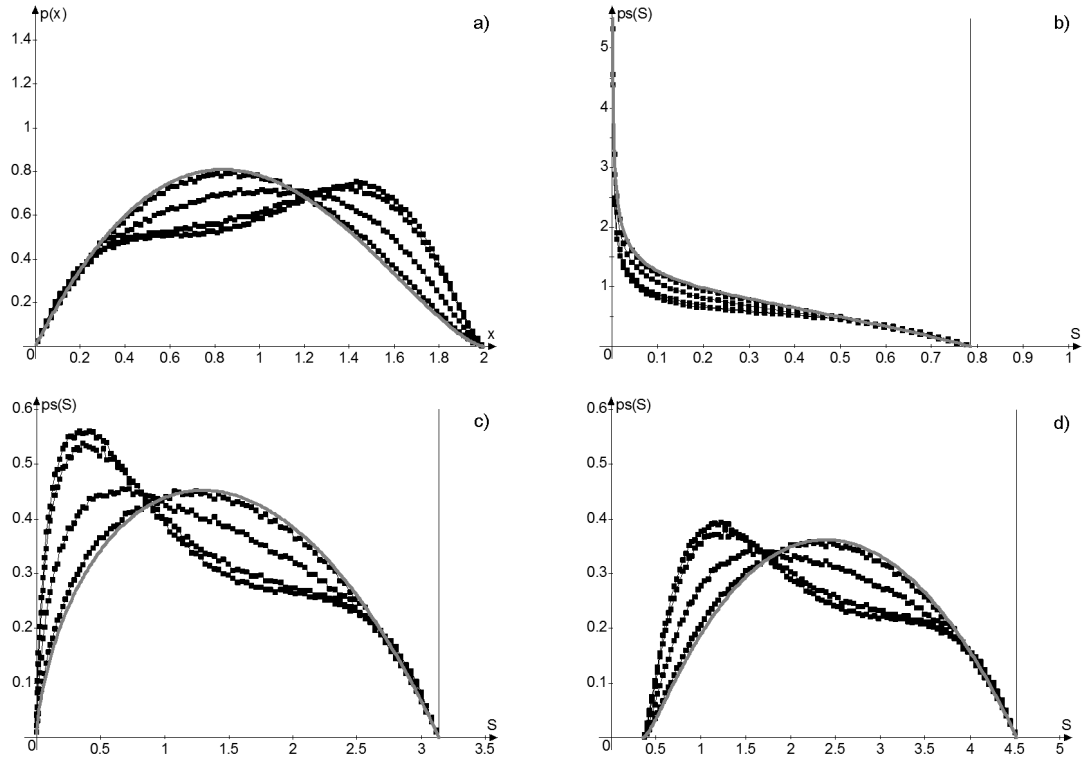

Fig. 6. The case of an "edge" parabolic inhomogeneity, $f(\rho) = A + \alpha\rho^2$, $h = 0$; 0.1; 0.5; 0.9.

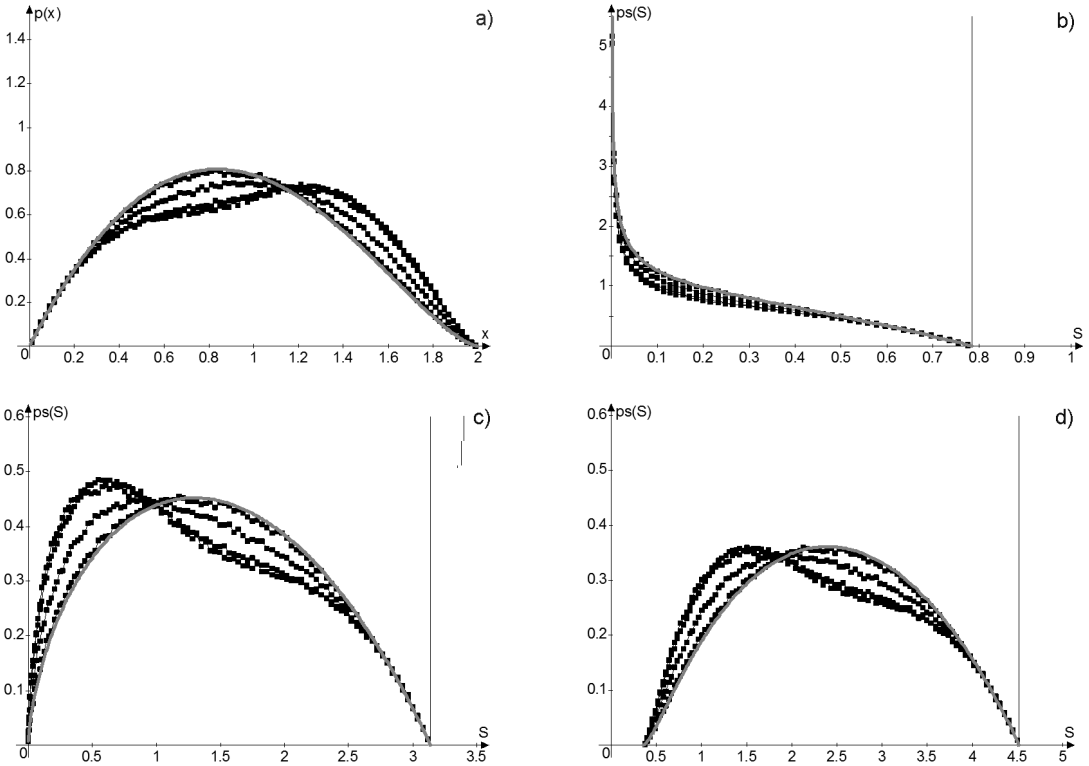

Fig. 7. The case of an "edge" linear inhomogeneity, $f(\rho) = A + \alpha\rho$, $h = 0$; 0.1; 0.5; 0.9.

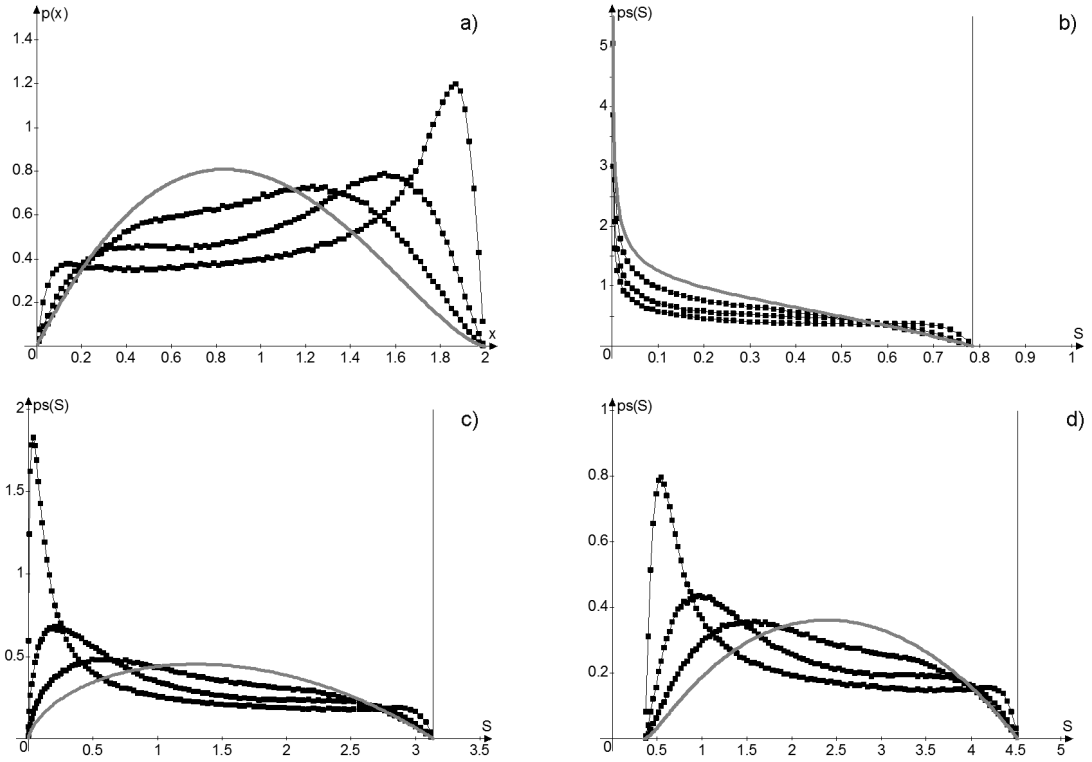

Fig. 8. The case of "edge" power-law inhomogeneity, $f(\rho) = A\rho^{\alpha}$, $r_w = 0.96$; 0.876; 0.79 (the r_w values are equivalent to $r_w = 0.1$; 0.3; 0.5 for the "central" inhomogeneity).

tioned above, in the cases of parabolic and linear "central" $f(\rho)$ functions, the largest inhomogeneity (the smallest $\rho_w = 0.5$) is obtained with linear $f(\rho)$ at $h = 0$.

For the Gaussian $f(\rho)$ distribution, we will select the parameters so as to obtain this ($\rho_w = 0.5$) and a stronger ($\rho_w < 0.5$) "center" inhomogeneity.

With "center" inhomogeneity with $\rho_w < 0.5$ realized under the Gaussian $f(\rho)$ distribution, a qualitative change in the type of density $p_s(S)$ at $r < R$ becomes noticeable (Fig. 5). Instead of a monotonically decreasing function, a function appears with a minimum and a subsequent maximum. The minimum is clearly visible on the plot for $\rho_w = 0.3$ (average graph), it is close to zero. This is a new kind of distribution density $p_s(S)$ of "windows", which is not observed in the homogeneous case. Transition to this type of $p_s(S)$ is observed in the graphs for $\rho_w = 0.5$, both with Gaussian and linear $f(\rho)$ functions, in the form of the appearance of a convex region, as mentioned above. The changes in $p(x)$ graphs and $p_s(S)$ graphs at $r = R$ and $r > R$ are qualiatively the same as in the cases above, but much stronger (pay attention to changing the scale of the vertical axis). Graphs for $r = R$ and $r > R$ very similar, the differences of the special case $r = R$ with such a strong central inhomogeneity are completely erased.

Let us turn to the cases of "edge" inhomogeneity (Fig. 6). As you can see, the behavior of the graphs with a parabolic "edge" inhomogeneity is more complicated than in the case of a parabolic "central" one. The general changes in the graphs, as expected, are the opposite of the changes for the "central" inhomogeneity: the graphs $p(x)$ are shifted to the right (the characteristic distance between the centers of the islands increases), while the graphs $p_s(S)$ are, respectively, shifted to the left ("windows" on, average, become smaller in area). But the shape of the graphs changes: a concave part appears between convex ones (except for the graph for $p_s(S)$ at $r < R$). This effect is due to the fact that the "central" inhomogeneity, although it reduces the effective radius within which the islands are formed, retains the circular shape of this region. The "edge" inhomogeneity changes the general shape of this region; it brings it closer to the shape of the ring at the edge of the layer.

As for the "central" case, the graph for $h = 0.9$ (closest to the "uniform" case with $h = 1$)

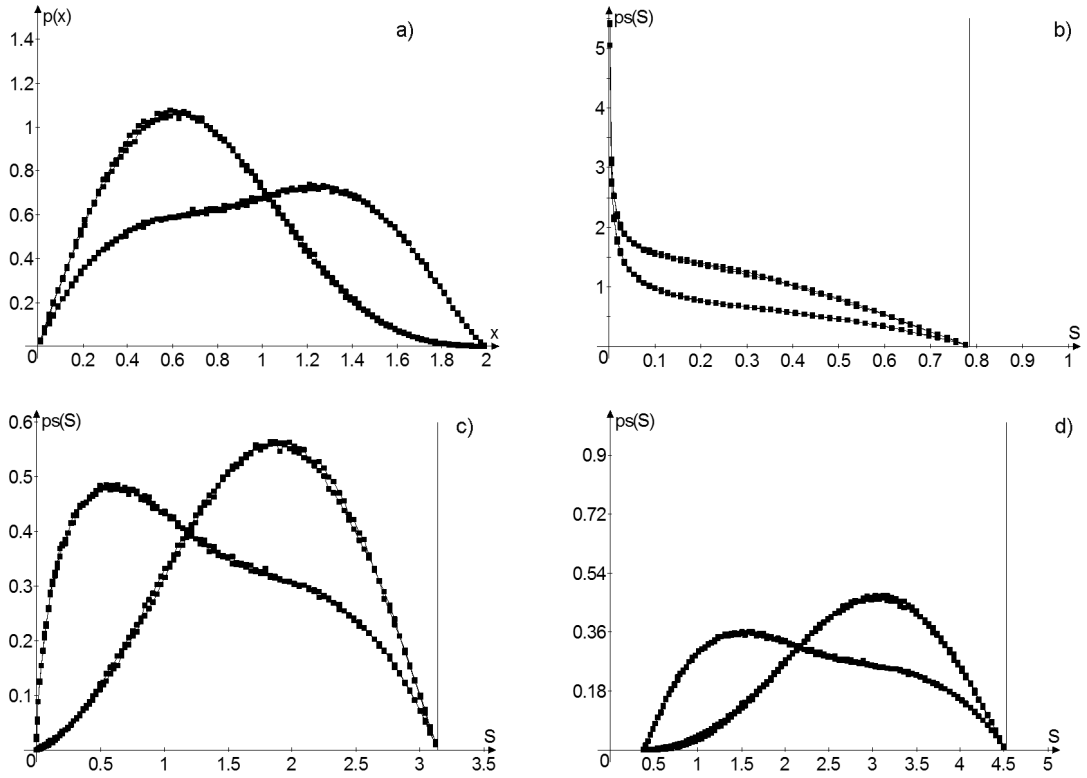

Fig. 9. Comparison of results with the same values ρ_w of inhomogeneity and different $f(\rho)$, for $\rho_w = 0.5$: "central" linear and Gaussian $f(\rho)$ for $\rho_w = 0.79$: "edge" linear and power $f(\rho)$.

is very close to the graph for the homogeneous case (gray continuous curve) obtained analytically in [1]. The graphs for $h = 0$ (maximum inhomogeneity) and $h = 0.9$ are close to each other, i.e. the dependence on h is almost absent; small inhomogeneity changes have little effect on the change in the graphs. (These graphs deviate as much as possible from the curve for a homogeneous case.) For the special case $r = R$, asymptotics to $S = 0$, apparently, remains convex with increasing inhomogeneity, as in the homogeneous case; although it is difficult to say exactly on the basis of the graph. Note, that with a small inhomogeneity (h = 0.5 and 0.9), the graph moves away from the homogeneous case, but retains its shape; with a further increase in the inhomogeneity, the shape also changes.

As in the case of the "central" inhomogeneity, the graphs for the linear "edge" inhomogeneity (Fig. 7) show the same behavior as for the parabolic case. However, in contrast to the "central" case, a change in the graph (relatively homogeneous case) for linear inhomogeneity is less than for parabolic with the same h. The parabolic "edge" inhomogeneity is stronger than the linear one with the same h.

Finally, we turn to the case of a power-law "edge" inhomogeneity (Fig. 8), which allows us to obtain arbitrarily large values of the inhomogeneity (ρ_w arbitrarily close to unity). The principle of selecting ρ_w to construct the graphs for the case of a power-law "edge" inhomogeneity is described above.

As in the case of the "central" inhomogeneity, the $p_s(S)$ graphs for $r > R$ and for a special case with a uniform distribution of islands ($r = R$) are similar to each other, except for a possible nonzero minimum of S_0 at $r > R$. With a significant "edge" inhomogeneity, a second maximum appears in the $p(x)$ graph, which is associated with the ring shape of the effective region of island nucleation. As a result, the second maximum appears on the $p_s(S)$ graphs at $r = R$ and $r > R$. It is also noticeable that, for the same reasons, the $p_s(S)$ graph for $r < R$ with a significant "edge" inhomogeneity intersects the graph for a homogeneous case. The same effect, but much less pronounced, can be seen in the graphs for a parabolic inhomogeneity.

Finally, let's compare the graphs $p(x)$ and $p_s(S)$ at the same ρ_w values of inhomogeneity and different $f(\rho)$ (Fig. 9). In accordance with the above, for comparison, we choose the "central" linear function $f(\rho) = A - \alpha\rho$ at h = 0 ($\rho_w = 0.5$) and the Gaussian $f(\rho) = A\exp(-\alpha\rho^2)$ at $\rho_w = 0.5$ for the "central" inhomogeneity; for the "edge" inhomogeneity, we use the "edge" linear function $f(\rho) = A + \alpha\rho$ at $h = 0$ ($\rho_w \approx 0.79$) and the power function at $f(\rho) = A\rho^{\alpha}$ at $\rho_w \approx 0.79$.

As you can see, the graphs in each pair are very close, almost coincide. This means that the behavior of the system, including the shape of $p_s(S)$ graphs, depends mainly on the value of the inhomogeneity, and much lesser on the specific type of the $f(\rho)$ function.

4. Conclusions

In this work, the formation of "windows" of transparency in a layered sample with layers consisting of a basic, opaque material and inclusions as islands of a transparent material was studied for the case when the islands are inhomogeneously distributed over the layer. We studied both the cases of a "central" inhomogeneity (the increasing probability of nucleation of islands closer to the center in comparison with the homogeneous case) and the cases of an "edge" inhomogeneity. The centrally symmetric inhomogeneity was considered. The radius ρ_w was chosen as a numerical measure of the inhomogeneity, such that the probability of the formation of an island within it is equal to 0.5 (and outside, respectively, also 0.5). We studied the cases of inhomogeneity with ρ_w from 0.1 to 0.96 in fractions of the layer radius R, i.e. in a wide range of inhomogeneity.

For selected types of inhomogeneous distribution of islands in the layer, densities $p_s(S)$ of distribution of the "windows" formed in the sample over the area were obtained. For the special case of two layers with one island in the layer, the results were obtained analytically for the homogeneous case using the solution of the problem of the distribution of distances between two random points. This problem was solved analytically also for one case of inhomogeneity; for other cases, numerical solutions were carried out.

The following results were obtained.

Densities $p_s(S)$ of distribution of the "windows" for the considered special case of two layers with one island per layer were obtained from expression (1) relating $p_s(S)$ and density $p(x)$ of distribution of distances between two random points in a circle. Density $p(x)$ was obtained analytically for a parabolic inhomogeneity of the distribution of islands in the layer and numerically for all studied cases. The analytical result coincides with the numerical one for parabolic inhomogeneity (Fig. 2).

For this special case, the type of distribution density $p_s(S)$ is qualitatively different for different ratios of the island radius r and the radius R of the area of nucleation of islands in the layer: $r < R$, $r = R$ and $r > R$. Also, this type strongly depends on the value of the inhomogeneity. The graphs for each of these three cases and various types of inhomogeneity are shown in Figs. 3–8. The following was shown:

— At small values of the "central" inhomogeneity (parabolic and linear, Figs. 3–4), the density $p_s(S)$ does not change its shape, only shifts to larger S. The only qualitative change is the small-scale asymptotic behavior for the special case $r = R$, from convex to concave with increasing inhomogeneity to $\rho_w \approx 0.5$;

— In the case of a significant "central" inhomogeneity (Gaussian, Fig. 5), the density curve $p_s(S)$ for the case $r = R$ acquire the same shape as for $r > R$ ($r = R$ ceases to be a special case), and when $\rho_w \approx 0.3$ and less, the density curve $p_s(S)$ for the case $r < R$ qualitatively changes its shape. Instead of a monotonous decrease, a minimum and a subsequent maximum appear on the density graph. Such changes are associated with a decrease in the effective R for the "central" inhomogeneity due to the "contraction" of the centers of nucleation of the islands to the center of the layer;

— In the case of the "edge" inhomogeneity, a qualitative change in the shape of the $p_s(S)$ curve is observed for $r = R$ and $r > R$ even in the case of a small inhomogeneity (parabolic and linear, Figs. 6 and 7): a concave section appears on the graph after the maximum. Also, in general, these graphs, as well as the graph for $r < R$ at the "edge" inhomogeneity, shift towards smaller areas S of the "windows".

— In the case of a strong "edge" inhomogeneity (power law, Fig. 8), a minimum appears in the $p_s(S)$ graphs for $r = R$ and $r > R$. Also, two maxima appear on these graphs: in the region of small S — the one that was in the homogeneous case, only shifted, and the second, smaller in size, in the region of large S. This kind of change in the $p_s(S)$ shape is due to the fact that the effective region of island nucleation with a strong "edge" inhomogeneity is not a circle but approaches the shape of the ring at the edge of the layer. The shift of the first maximum towards smaller S is associated with the move

away, on average, of the islands from each other; the appearance of the second maximum is associated with a general decrease in the area of the effective region of nucleation of the islands with the strong "edge" inhomogeneity.

— From the results obtained it can be seen (Fig. 9) that for the same ρ_w value, but different types of inhomogeneity distribution function (for example, linear and Gaussian), the graph $p_s(S)$ looks the same, i.e. the shape of $p_s(S)$ depends mainly on the magnitude of the inhomogeneity, and not on its specific form. The same can be seen from a comparison of the graphs for linear and parabolic inhomogeneity, both of the "central" and the "edge" ones.

References

1. R.Brodskii, A.V.Tur, V.V.Yanovsky, *Functional Materials*, **21**, 448 (2014).
2. R.Ye.Brodskii, *Functional Materials*, **27**, 159 (2019).
3. R.Ye.Brodskii, *Functional Materials*, **24**, 91 (2017).
4. R.Ye.Brodskii, *Functional Materials*, **25**, 780 (2018).
5. R.Ye.Brodskii, *Functional Materials*, **26**, 587 (2019).
6. M.Kendall, P.Moran, Geometric Probabilities, Nauka, Moscow (1972) [in Russian].